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Extremal functions for singular Moser–Trudinger embeddings

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ABSTRACT

We study Moser–Trudinger type functionals in the presence of singular potentials. In particular we propose a proof of a singular Carleson–Chang type estimate by means of Onofri's inequality for the unit disk in $\mathbb{R}^2$. Moreover we extend the analysis of Adimurthi (2004) and Csató and Roy (2015a) considering Adimurthi–Druet type functionals on compact surfaces with conical singularities and discussing the existence of extremals for such functionals.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain, from the well known Sobolev's inequality

$$\|u\|_{L^{\frac{2p}{2-p}}(\Omega)} \le S_p \|\nabla u\|_{L^p(\Omega)} \quad p \in (1,2),\ u \in W_0^{1,p}(\Omega), \tag{1}$$

one can deduce that the Sobolev space $H_0^1(\Omega) := W_0^{1,2}(\Omega)$ is embedded into $L^q(\Omega)$ $\forall\ q \ge 1$. A much more precise result was proved in 1967 by Trudinger [27]: on bounded subsets of $H_0^1(\Omega)$ one has uniform exponential-type integrability. Specifically, there exists $\beta > 0$ such that

$$\sup_{u \in H_0^1(\Omega),\ \int_\Omega |\nabla u|^2 dx \le 1} \int_\Omega e^{\beta u^2} dx < +\infty. \tag{2}$$

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This inequality was later improved by Moser in [20], who proved that the sharp exponent in (2) is $\beta = 4\pi$, that is

$$\sup_{u\in H_0^1(\Omega),\ \int_\Omega |\nabla u|^2 dx\le 1} \int_\Omega e^{4\pi u^2} dx < +\infty, \tag{3}$$

and

$$\sup_{u\in H_0^1(\Omega),\ \int_\Omega |\nabla u|^2 dx\le 1} \int_\Omega e^{\beta u^2} dx = +\infty \tag{4}$$

for $\beta > 4\pi$. An interesting question consists in studying the existence of extremal functions for (3). Indeed, while there is no function realizing equality in (1), one can prove that the supremum in (3) is always attained. This was proved in [4] by Carleson and Chang for the unit disk $D \subseteq \mathbb{R}^2$, and by Flucher [9] for arbitrary bounded domains (see also [23,15]). The proof of these results is based on a concentration-compactness alternative stated by P. L. Lions [16]: for a sequence $u_n \in H_0^1(\Omega)$ such that $\|\nabla u_n\|_{L^2(\Omega)} = 1$ one has, up to subsequences, either

$$\int_\Omega e^{4\pi u_n^2} dx \to \int_\Omega e^{4\pi u^2} dx,$$

where u is the weak limit of u_n, or u_n concentrates at a point $x \in \overline{\Omega}$, that is

$$|\nabla u|^2 dx \rightharpoonup \delta_x \quad \text{and} \quad u_n \rightharpoonup 0. \tag{5}$$

The key step in [4] consists in proving that if a sequence of radially symmetric functions $u_n \in H_0^1(D)$ concentrates at 0, then

$$\limsup_{n\to\infty} \int_D e^{4\pi u_n^2} dx \le \pi(1+e). \tag{6}$$

Since for the unit disk the supremum in (3) is strictly greater than $\pi(1+e)$, one can exclude concentration for maximizing sequences by means of (6), and prove existence of extremal functions for (3). In [9] Flucher observed that concentration at arbitrary points of a general domain Ω can always be reduced, through properly defined rearrangements, to concentration of radially symmetric functions on the unit disk. In particular he proved that if $u_n \in H_0^1(\Omega)$ satisfies $\|\nabla u_n\|_2 = 1$ and (5), then

$$\limsup_{n\to\infty} \int_\Omega e^{4\pi u_n^2} dx \le \pi e^{1+4\pi A_\Omega(x)} + |\Omega|, \tag{7}$$

where $A_\Omega(x)$ is the Robin function of Ω, that is the trace of the regular part of the Green function of Ω. He also proved

$$\sup_{u\in H_0^1(\Omega), \int_\Omega |\nabla u|^2 dx\le 1} \int_\Omega e^{4\pi u^2} dx > \pi e^{1+4\pi \max_{\overline{\Omega}} A_\Omega} + |\Omega|,$$

which implies the existence of extremals for (3) on Ω. Similar results hold if Ω is replaced by a smooth closed surface (Σ, g). Let us denote

$$\mathcal{H} := \left\{ u \in H^1(\Sigma) : \int_\Sigma |\nabla u|^2 dv_g \le 1,\ \int_\Sigma u\, dv_g = 0 \right\}.$$

Fontana [10] proved that

$$\sup_{u\in\mathcal{H}} \int_\Sigma e^{4\pi u^2} dv_g < +\infty, \tag{8}$$