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Blow-up solutions for the modified b-family of equations

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ABSTRACT

In this paper, we firstly derive the blow-up criterion and the lower bound of the maximal time of existence to the modified b-family of equations. The new blow-up mechanism for the solutions with certain initial profiles and various parameters is then described in detail.

1. Introduction

This paper is concerned with the Cauchy problem for the following modified b-family of equations

$$\begin{cases} m_t + k_1((u^2 - u_x^2)m)_x + k_2(bu_x m + um_x) + \gamma u_x = 0, \\ u(0,x) = u_0(x), \quad t > 0, x \in R, \end{cases} \tag{1.1}$$

where

$$m = u - u_{xx}, \tag{1.2}$$

represents the *momentum density* of the system, b is a real parameter and $\gamma \in R$ characterizes the effect of the linear dispersion. Eq. (1.1) reduced to the modified Camassa–Holm equation [41,42]

$$m_t + ((u^2 - u_x^2)m)_x + \gamma u_x = 0,$$

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for $k_1 = 1, k_2 = 0$, the b-family of equations [31,32]

$$m_t + um_x + bu_x m + \gamma u_x = 0, \quad m = u - u_{xx},$$

for $k_1 = 0, k_2 = 1$, respectively.

The original Camassa–Holm (CH) [2,3] equation

$$m_t + um_x + 2u_x m + \gamma u_x = 0,$$

where m is as above, (1.2), can itself be derived from Korteweg–deVries (KdV) equation by tri-Hamiltonian duality. The CH equation [16,26] was proposed as a model for surface waves, and has been studied extensively because of its many remarkable properties: infinity of conservation laws and complete integrability [2,26], with action angle variable constructed using inverse scattering [13,17], existence of peaked solitons and multi-peakons [2], geometric formulations [15,33], well-posedness and breaking waves, meaning solutions that remain bounded while its slope becomes unbounded in finite time [5,10,11]. The well-posedness of the CH equation has been shown in [30,37] with the initial data $u_0 \in H^s(R), s > \frac{3}{2}$. Moreover, if the initial momentum density $m_0(x) = m(0, x)$ does not change sign, the Cauchy problem admits global solution for certain initial values [5,8,10], whereas solutions may blow-up if their initial momentum density m_0 changes sign [5,8–10]. When $\gamma = 0$, the b-family equations for $b = 2$ particularize to the celebrated Degasperis–Procesi (DP) equation [21], an integrable hydrodynamical model with peaked traveling waves and with breaking wave solutions [14], which takes the form

$$m_t + um_x + 3u_x m = 0, \quad m = u - u_{xx}.$$

Note that the CH and DP equations peakons are orbitally stable—their shape is stable under small perturbations and thus these wave patterns are detectable [18,34–36]. This is relevant because the traveling wave solutions of greatest height of the governing equations for water waves are peaked [6,7,12,39]. The geometric interpretation, infinite propagation speed and the inverse scattering transform for the DP equation were studied recently in [14,23,29].

Like the KdV equation, the CH equation has quadratic nonlinear terms. It is of great interest to find such integrable equations with cubic or higher nonlinear terms. Two CH type equations with cubic nonlinearities have been proposed: the modified Camassa–Holm (mCH) equation, [25,41,42]

$$m_t + ((u^2 - u_x^2)m)_x = 0, \quad m = u - u_{xx}, \tag{1.3}$$

and the Novikov equation [40]

$$m_t + u^2 m_x + \frac{3}{2}(u^2)_x m = 0. \tag{1.4}$$

Both equations have peaked solitons and can be used to model wave breaking. The mCH equation can be derived by applying the method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the focusing modified Korteweg–de Vries (mKdv) equation [41]. It was shown that the mCH equation has a single peaked soliton and multi-peaks with a different character than those of the CH equation [27], and it also has new features of blow-up criterion and wave breaking mechanism. As an extension of both the CH and mCH equations, an integrable equation with both quadratic and cubic nonlinearities has been introduced by Fokas [25], which takes the form

$$m_t + k_1((u^2 - u_x^2)m)_x + k_2(2u_x m + um_x) + \gamma u_x = 0, \quad m = u - u_{xx}. \tag{1.5}$$

There is a distinctive feature for gmCH equation is that it is a integrable model for the breakdown of regularity. Moreover, it admits a remarkable variety of the so-called “peakon” solutions, which means wave solutions with a discontinuous derivative at crest. Physically, those peakons reveal some similarity to the well-known Stokes waves of greatest height-the traveling waves of maximum possible amplitude that a resolutions