Nonlinear Analysis 150 (2017) 38–60

Contents lists available at ScienceDirect

Nonlinear Analysis

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2D Quasi-Geostrophic equation; sub-critical and critical cases

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ARTICLE INFO

Article history:
Received 23 September 2016
Accepted 3 November 2016
Communicated by Enzo Mitidieri

MSC:
35Q30
76D05
35S15
35A25

Keywords:
Quasi-Geostrophic equation
Local and global solvability
Critical equation

ABSTRACT

Our task here is to use a version of the 'vanishing viscosity technique' to study the *critical 2D Quasi-Geostrophic equation*. The present paper extends and specializes the results reported in Dlotko et al. (2015). We treat now in more detail the solutions of the critical problem ($\alpha = \frac{1}{2}$); in particular their uniqueness, regularity and long time behavior.

1. Introduction

We consider the Cauchy and Dirichlet problems for sub-critical and critical *Viscous Surface Quasi-Geostrophic Equation* (Q–G equation, for short) [4,9,13,17,25,31,41,42]:

$$\begin{aligned} &\theta_t + u\cdot\nabla\theta + \kappa(-\Delta)^{\alpha}\theta = f, \quad x\in\Omega,\, t>0,\\ &\theta(0,x) = \theta_0(x), \end{aligned} \tag{1.1}$$

where θ represents the potential temperature, $\kappa > 0$ is a diffusivity coefficient, $\alpha \in [\frac{1}{2}, 1]$ a fractional exponent, and $u = (u_1, u_2)$ is the *velocity field* determined by θ through the relation:

$$u = \Big(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\Big), \quad \text{where } (-\Delta)^{\frac{1}{2}}\psi = -\theta \tag{1.2}$$

or, in a more explicit way,

$$u = \mathcal{R}^{\perp}\theta \quad \text{with } \mathcal{R} = \nabla(-\Delta)^{-\frac{1}{2}}. \tag{1.3}$$

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http://dx.doi.org/10.1016/j.na.2016.11.005

We consider here both the Dirichlet boundary value problem in a bounded regular (e.g., of the class $C^{2,\gamma}$) domain $\Omega \subset \mathbb{R}^2$ and simultaneously the Cauchy problem with $\Omega = \mathbb{R}^2$.

For the whole $\mathbb{R}^2$, the fractional Laplacian was defined first using Fourier transform (e.g., see [37]);

$$\widehat{(-\Delta)^{\alpha}} v(\xi) = |\xi|^{2\alpha} \hat{v}(\xi). \tag{1.4}$$

For the bounded smooth domain case, we use the following definition of Balakrishnan/Komatsu (e.g., see [3,10,26,27]):

$$(-\Delta)^{\alpha} g = \frac{\sin(\alpha\pi)}{\pi} \int_0^{\infty} s^{\alpha-1}(sI - \Delta)^{-1}(-\Delta) g ds, \quad g \in D(-\Delta),\ \alpha \in (0,1). \tag{1.5}$$

Equivalence of several definitions of the fractional minus Laplace operator in $\mathbb{R}^N$ (including the Balakrishnan, the singular integral and the Fourier multiplier definitions) was reported recently in [28, Theorem 5.7]. This equivalence is true in particular for $\alpha \in (0,1)$ and operators settled in $L^p(\mathbb{R}^N), p \in [1,\infty)$, $C_0(\mathbb{R}^N)$ (the space of continuous functions vanishing at infinity), or the space $C_{bu}(\mathbb{R}^N)$ (the space of bounded and uniformly continuous functions).

Extending our technique of [17], as in the papers devoted to the Navier–Stokes equations [15,7,21], we analyze first the sub-critical case of exponents $\alpha \in (\frac{1}{2}, 1]$, letting then the parameter $\alpha \to \frac{1}{2}^+$. See [31] for another regularization of the problem, where an extra Laplacian is added to join the existing dissipative term (see also [13, p. 524]). We remark here that in the sub-critical case $\alpha \in (\frac{1}{2}, 1]$ the already existing dissipation $\kappa(-\Delta)^{\alpha}\theta$ alone is strong enough to guarantee good properties of solutions; the regularization is needed when $\alpha \in [0, \frac{1}{2}]$.

The technique used in the present and earlier publications [7,17,15] is a variant of the classical *vanishing viscosity technique* that comes back to the 1950s and the studies of E. Hopf, O.A. Oleinik, P.D. Lax and J.-L. Lions dealing initially with the Burgers model and some problems in gas dynamics (e.g. [35]). It is applicable to critical and super-critical equations, in which the nonlinear term is 'equivalent or more valid' than the main dissipative term in the equation (equivalent with first or higher power of the main part operator). Our idea is simple. We just strengthen that dissipative term, replacing with its fractional power with sufficiently large exponent greater than 1. Next, using the strong and elegant semigroup technique we solve easily the modified problems. The final step is to pass to the limit over a sequence of solutions to such regularized problems, to obtain a weak solution of the original problem. Essential in that step are the *uniform with respect to the approximation parameter* estimates of solutions to the regularized problems. Such technique applies, among others, to the 3D Navier–Stokes equation (N–S equation, for short), and the results obtained [7] are fully comparable with those known in the literature, obtained within another approaches. The advantage here is that the approximations we are using are smooth (since they are solutions of the regular dissipative problems). Moreover, the approximating problems are very similar in nature to their critical or super-critical limit. Note that for regular functions the difference between $(-\Delta)\phi$ and $(-\Delta)^{1+\epsilon}\phi$ tends to zero as $\epsilon \to 0^+$. The solutions to our approximations exist globally in time, while for the limits they may be only local, with difficult estimation of the life time. This property helps when looking numerically for solutions.

Let us discuss briefly the difference, in the light of the above described technique, between the Navier–Stokes equation (in 3-D, say) and the 2-D viscous Quasi-Geostrophic equation. That difference is significant, since the N–S equation is *locally well posed* in many phase spaces used in considerations of that problem. This property is connected with the fact that the nonlinearity of the (even 3-D) N–S acts between spaces of the fractional order scale corresponding to the Stokes operator, the difference of which (measured by the difference of exponents) is strictly less than one. Consequently, we will use directly the standard semigroup approach (like in [33,22,6]) to semilinear problems to get *local in time* solvability. The weakness in case of the N–S equation is that the known in 3-D a priori estimates are too weak to guarantee,