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Fractal Strichartz estimate for the wave equation

Chu-Hee Cho[a], Seheon Ham[b,*], Sanghyuk Lee[a]

[a] School of Mathematical Sciences, Seoul National University, Seoul 08826, Republic of Korea
[b] School of Mathematics, Korea Institute for Advanced Study, Seoul 02455, Republic of Korea

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ABSTRACT

We consider Strichartz estimates for the wave equation with respect to general measures which satisfy certain growth conditions. In $\mathbb{R}^{3+1}$ we obtain the sharp estimate and in higher dimensions improve the previous results.

1. Introduction

Let us consider the wave equation in $\mathbb{R}^n \times \mathbb{R}$:

$$\begin{cases} \partial_t^2 u - \Delta u = 0, \\ u(x,0) = f, \qquad \partial_t u(x,0) = g. \end{cases} \tag{1}$$

The space–time estimate for the solution of (1) which is called *Strichartz estimate* has been proven to be an important tool in studies of various problems. (See [22,21,12,14,18,11].) It is well-known that the estimate

$$\|u\|_{L_t^q(\mathbb{R},\, L_x^r(\mathbb{R}^n))} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} \tag{2}$$

holds for $s \ge 0, 2 \le q, r < \infty$ which satisfy

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s, \qquad \frac{1}{q} + \frac{n-1}{2r} \le \frac{n-1}{4}.$$

Here $\dot{H}^s$ is the homogeneous L^2 Sobolev space of order s. See [7] for the estimates with $r = \infty$. It was Strichartz [22] who first proved the estimate (2) when $q = r$. This was later extended to mixed norm estimates by Pecher [21]. (Also see [8].) The endpoint cases $(r, q) = (2(n-1)/(n-3), 2)$ except $n = 3$ were obtained by Keel–Tao [11]. Klainerman and Machedon [12] showed the failure of (2) when $(n, r, q) = (3, \infty, 2)$.

* Corresponding author.
E-mail addresses: akilus@snu.ac.kr (C.-H. Cho), hamsh@kias.re.kr (S. Ham), shklee@snu.ac.kr (S. Lee).

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In this note we consider a generalization of (2) by replacing the Lebesgue measure with general measure μ. More precisely, we study the estimate

$$\|u\|_{L^q(d\mu)} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}}. \tag{3}$$

Here we denote by $H^s(\mathbb{R}^n)$ the inhomogeneous L^2 Sobolev space of order s, which is the space of all tempered distributions f such that $(1+|\cdot|^2)^{\frac{s}{2}}\widehat{f} \in L^2(\mathbb{R}^n)$, equipped with the norm

$$\|f\|_{H^s(\mathbb{R}^n)} = \|(1+|\cdot|^2)^{\frac{s}{2}}\widehat{f}\|_{L^2(\mathbb{R}^n)}.$$

This kind of estimates was studied in connection with problems in geometric measure theory, precisely, the sphere packing problem (see [17,27,19,20]).

Throughout this paper, the measure μ is assumed to be a nonnegative Borel regular measure with compact support in $\mathbb{R}^{n+1}$. Let us denote by $\mathfrak{M}(\mathbb{R}^{n+1})$ the space of such measures. In addition we impose uniform growth condition on μ as follows.

Definition 1.1. Let $\alpha \in (0, n+1]$. For $\mu \in \mathfrak{M}(\mathbb{R}^{n+1})$, we say that μ is α-dimensional if there exists a constant C_μ, independent of x and ρ, such that

$$\mu(B(x,\rho)) \le C_\mu \rho^\alpha \quad \text{for all } x \in \mathbb{R}^{n+1},\ \rho > 0. \tag{4}$$

Here $B(x,\rho)$ denotes the open ball of radius ρ centered at x. Also we define

$$\langle \mu \rangle_\alpha = \sup_{x\in\mathbb{R}^{n+1},\, \rho>0} \rho^{-\alpha}\mu(B(x,\rho)).$$

For $1 \le q \le \infty$ let us set

$$s(\alpha,q,n) = \begin{cases} \max\left\{\frac{n}{2}-\frac{\alpha}{q}, \frac{n+1}{4}\right\}, & \text{if } 0<\alpha\le 1,\\ \max\left\{\frac{n}{2}-\frac{\alpha}{q}, \frac{n+1}{4}+\frac{1-\alpha}{2q}, \frac{n+2}{4}-\frac{\alpha}{4}\right\}, & \text{if } 1<\alpha\le n,\\ \max\left\{\frac{n}{2}-\frac{\alpha}{q}, \frac{n+1}{4}+\frac{n+1-2\alpha}{2q}, \frac{n+1}{2}-\frac{\alpha}{2}\right\}, & \text{if } n<\alpha\le n+1. \end{cases} \tag{5}$$

When $n=2$ Wolff [27] showed that (3) holds for α-dimensional measure μ if $s > \max(\frac{3}{4}, 1-\frac{\alpha}{4}, 1-\frac{\alpha}{q})$, $\alpha \in (1,3)$. Erdoğan [4] improved Wolff's result so that (3) holds for $s > s(\alpha,q,2)$, $\alpha \in (1,3)$ and also showed that (3) generally fails if $s < s(\alpha,q,2)$. When $n \ge 3$, Oberlin [19] obtained (3) for $\alpha \in (1,n+1)$ provided that $q < \alpha$ and $s > \frac{n-1}{2}$.

It is plausible to conjecture that (3) holds if $s > s(\alpha,q,n)$ (see Proposition 1.5) but like other open problems of similar nature complete resolution seems out of reach at this moment. However, for $n=3$ and $\alpha \in [1,4]$, we obtain the sharp estimate by the following theorem and Proposition 1.5.

Theorem 1.2. *Let $n=3$. Also let μ be an α-dimensional measure. Suppose that u is a solution to Eq. (1). Then (3) holds with*

$$s > \begin{cases} s(\alpha,q,3), & \text{if } 2 \le q \le \infty,\\ s(\alpha,2,3), & \text{if } 1 \le q \le 2. \end{cases} \tag{6}$$

Furthermore, the implicit constant in (3) does not depend on particular choice of μ as long as $\langle\mu\rangle_\alpha$ is uniformly bounded.