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p-harmonic ℓ-forms on Riemannian manifolds with a weighted Poincaré inequality

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ABSTRACT

Given a Riemannian manifold with a weighted Poincaré inequality, in this paper, we will show some vanishing type theorems for p-harmonic ℓ-forms on such a manifold. We also prove a vanishing result on submanifolds in Euclidean space with flat normal bundle. Our results can be considered as generalizations of the work of Lam, Li–Wang, Lin, and Vieira (see Lam (2008), Li and Wang (2001), Lin (2015), Vieira (2016)). Moreover, we also prove a vanishing and splitting type theorem for p-harmonic functions on manifolds with Spin (9) holonomy provided a (p, p, λ)-Sobolev type inequality which can be considered as a general Poincaré inequality holds true.

1. Introduction

Let (M^n, g) be a Riemannian manifold of dimension n and $\rho \in \mathcal{C}(M)$ be a positive function on M. We say that M has a weighted Poincaré inequality, if

$$\int_M \rho\varphi^2 \leq \int_M |\nabla\varphi|^2 \tag{1.1}$$

holds true for any smooth function $\varphi \in \mathcal{C}_0^\infty(M)$ with compact support in M. The positive function ρ is called the weighted function. It is easy to see that if the bottom of the spectrum of Laplacian $\lambda_1(M)$ is positive then M satisfies a weighted Poincaré inequality with $\rho \equiv \lambda_1$. Here $\lambda_1(M)$ can be characterized by variational principle

$$\lambda_1(M) = \inf\left\{\frac{\int_M |\nabla\varphi|^2}{\int_M \varphi^2} : \varphi \in \mathcal{C}_0^\infty(M)\right\}.$$

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When M satisfies a weighted Poincaré inequality then M has many interesting properties concerning topology and geometry. It is worth to notice that weighted Poincaré inequalities not only generalize the first eigenvalue of the Laplacian, but also appear naturally in other PDE and geometric problems. For example, $\lambda_1(M)$ is related to the problem of finding the best constant in the inequality

$$\|u\|_{L^2} \leq C\|\nabla u\|_{L^2}$$

obtained by the continuous embedding $W_0^{1,2} \to L^2(M)$. It is also well known that a stable minimal hypersurface satisfies a weighted Poincaré inequality with the weight function

$$\rho = |A|^2 + Ric(\nu, \nu)$$

where A is the second fundamental form and $Ric(\nu, \nu)$ is the Ricci curvature of the ambient space in the normal direction. For further discussion on this topic, we refer to [12,15,18,20,23] and the references there in.

On the other hand, suppose that M is a complete noncompact oriented Riemannian manifold of dimension n. At a point $x \in M$, let $\{\omega_1, \ldots, \omega_n\}$ be a positively oriented orthonormal coframe on $T_x^*(M)$, for $\ell \geq 1$, the Hodge star operator is given by

$$*(\omega_{i_1} \wedge \cdots \wedge \omega_{i_\ell}) = \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-\ell}},$$

where $j_1, \ldots, j_{n-\ell}$ is selected such that $\{\omega_{i_1}, \ldots, \omega_{i_\ell}, \omega_{j_1}, \ldots, \omega_{j_{n-\ell}}\}$ gives a positive orientation. Let d is the exterior differential operator, so its dual operator δ is defined by

$$\delta = *d * .$$

Then the Hodge–Laplace–Beltrami operator Δ acting on the space of smooth ℓ-forms $\Omega^\ell(M)$ is of form

$$\Delta = -(\delta d + d\delta).$$

In [15], Li studied Sobolev inequality on spaces of harmonic ℓ-forms. Then he gave estimates of the bottom of ℓ-spectrum and proved that the space of harmonic ℓ-forms is of finite dimension provided the Ricci curvature bounded from below. When M is compact, it is well-known that the space of harmonic ℓ-forms is isomorphic to its ℓ-th de Rham cohomology group. This is not true when M is non-compact but the theory of L^2 harmonic forms still has some interesting applications. For further results, one can refer to [4,5]. Later, in [20], the authors investigated spaces of L^2 harmonic ℓ-forms $H^\ell(L^2(M))$ on submanifolds in Euclidean space with flat normal bundle. Assuming that the submanifolds are of finite total curvature, Lin showed that the space $H^\ell(L^2(M))$ has finite dimension. Recently, in [23], Vieira obtained vanishing theorems for L^2 harmonic 1 forms on complete Riemannian manifolds satisfying a weighted Poincaré inequality and having a certain lower bound of the curvature. His theorems improve Li–Wang's and Lam's results. Moreover, some applications to study geometric structures of minimal hypersurfaces are also given. Therefore, it is very natural for us to study p-harmonic ℓ-forms on Riemannian manifolds with a weighted Poincaré inequality.

Recall that an ℓ-form ω on M is said to be p-harmonic ($p > 1$) if ω satisfies $d\omega = 0$ and $\delta(|\omega|^{p-2}\omega) = 0$. When $p = 2$, a p-harmonic ℓ-form is exactly a harmonic ℓ-form. Some properties of the space of p-harmonic ℓ-forms are given by X. Zhang and Chang–Guo–Sung (see [24,7]). In particular, in [6], Chang–Chen–Wei studied p-harmonic functions with finite L^q energy and proved some vanishing type theorems on Riemannian manifold satisfying a weighted Poincaré inequality, recently. Moreover, Sung–Wang, Dat and the author used theory of p-harmonic functions to show some interesting rigidity properties of Riemannian manifolds with maximal p-spectrum. (See also [8,22]).