Nonlinear Analysis 150 (2017) 159–193

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Small-amplitude solitary and generalized solitary traveling waves in a gravity two-layer fluid with vorticity

Ling-Jun Wang

College of Science, Wuhan University of Science and Technology, 430065 Wuhan, China

ARTICLE INFO

Article history:
Received 14 July 2016
Accepted 18 November 2016
Communicated by Enzo Mitidieri

MSC:
35J65
35Q35
76B03
76B15

Keywords:
Two-layer fluid
Vorticity
Gravity
Bifurcation theory

ABSTRACT

We prove existence of small-amplitude solitary and generalized solitary gravity waves traveling at the interface and the free surface of a two-dimensional two-layer fluid with vorticity and finite thickness. In both layers of the fluid the horizontal velocity of the fluid particle does not exceed the wave speed throughout the domain. The arguments are based on a formulation of the hydrodynamic problem as an infinite-dimensional Hamiltonian system in which the horizontal spatial direction is the timelike variable. A center-manifold reduction technique and a variety of dynamical systems methods are employed to detect homoclinic solutions to 0 and homoclinic solutions to periodic orbits to the reduced system, which correspond respectively to solitary and generalized solitary water waves.

1. Introduction

Consider a fluid with two immiscible layers, which is assumed to be over the rigid flat bed $\tilde{y} = -d$ with $0 < d < \infty$, bounded above by the free surface $\tilde{y} = \ell + f(\tilde{x}, t)$ with $0 < \ell < \infty$, and partitioned by the interface $\tilde{y} = \eta(\tilde{x}, t)$, so that the bottom layer and the upper layer of the fluid occupy at a given time t respectively the domains

$$\tilde{D}_1(t) = \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 : -d < \tilde{y} < \eta(\tilde{x}, t)\} \quad \text{and} \quad \tilde{D}_2(t) = \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 : \eta(\tilde{x}, t) < \tilde{y} < \ell + f(\tilde{x}, t)\}.$$

Both the thickness at rest of the bottom layer and upper layer are finite; it is d for the bottom layer, and ℓ for the upper layer. We assume that the fluid is perfect with constant densities ρ_1 in the bottom layer and ρ_2 in the upper layer; the fluid is acted upon by gravity and there is no surface tension, neither at the free surface nor at the interface; moreover, the fluid is rotational with vorticities $\omega_1(\tilde{x}, \tilde{y}, t)$ and $\omega_2(\tilde{x}, \tilde{y}, t)$ respectively in the bottom and upper layer. We are interested in traveling waves of horizontal velocity $c > 0$ at the interface

E-mail address: wanglingjun@wust.edu.cn.

http://dx.doi.org/10.1016/j.na.2016.11.012

and the free surface so that $\eta(\tilde{x},t)$ and $f(\tilde{x},t)$ exhibit a $(\tilde{x},t)$-dependence of the form $\tilde{x}-ct$. In the frame of reference moving with the wave, which is equivalent to the change of variables $(\tilde{x}-ct,\tilde{y})\mapsto(x,y)$, we use $D=D_1\cup D_2$ to denote the stationary domain the fluid occupies, where

$$D_1=\{(x,y)\in\mathbb{R}^2:-d<y<\eta(x)\}\quad\text{and}\quad D_2=\{(x,y)\in\mathbb{R}^2:\eta(x)<y<\ell+f(x)\}$$

are respectively the bottom region and the upper region. Finally, as in some studies of the rotational waves, we require that the horizontal velocity of the fluid particle in both layers does not exceed the wave speed, and exclude so the presence of stagnation points, where fluid particles move at the wave speed c and thus has zero velocity relative to the steady flow. Under this regime, we construct in this paper existence theories for solitary and generalized solitary waves traveling at the interface and the surface of the two-layer fluid. Here solitary interfacial (surface) waves are the waves whose profile $\eta(x)$ $(f(x))$ decays to zero at infinity, that is, $\eta(x)\to 0$ $(f(x)\to 0)$ as $x\to\pm\infty$, while generalized solitary waves are those whose pulselike profile decays at infinity, instead of to zero, to a periodic ripple whose amplitude is exponentially small compared to that of the pulse.

The water-wave problem is mainly studied in the regime of constant (homogeneous) and stratified (heterogeneous) density distribution. The two-layer fluid is one of the two principle types of stratified fluids, the other type of which is the fluid being continuously stratified in the sense that the density of the fluid is at least continuously regular or further smooth throughout the fluid domain. The bottom of the two-layer fluid we consider here is rigid while the surface of the upper layer is free, which is similar as the two-layer fluid studied for instance in Dias and Iooss [17], Sun [37], and Iooss et al. [25,28] where the depth of the bottom layer is however infinite. The free upper surface complicates the governing equations by introducing a fully-nonlinear boundary condition. We point out that such two-layer fluid flow with a free surface, a free interface and a flat rigid bed is typical for equatorial ocean waves; see, for instance, Constantin et al. [10,11,14,24]. Also a large part of the study of waves for two-layer fluids focused on flows whose upper and bottom boundary are both rigid. For such type of flows with two rigid boundaries, we refer to e.g., Amick and Turner [2], Sun [38] with however surface tension on the interface, Amick [1] and Sun [36] with infinite-depth bottom layer, Compelli et al. [9,8].

Most of the above works are concerned with irrotational waves. However, rotational traveling water waves have been extensively studied since the breakthrough work of Constantin and Strauss [16]; see for instance [32,5] for rotational *periodic* waves in two-layer systems with or without stagnations, and [39–41] for rotational large-amplitude *periodic continuously stratified* waves respectively for vanishing and nonvanishing surface tension. In the present work we intend to construct small-amplitude *solitary* and *generalized solitary* waves for the rotational two-layer fluid. It is worth mentioning that recently R. B. Chen et al. [7,6] established large-amplitude gravity solitary waves moving at the surface of a body of continuously stratified water, in the absence of stagnation. The small-amplitude solitary and generalized solitary waves result here may be used as a foundation for the construction of the corresponding large-amplitude waves. We also note that a general Hamiltonian framework for wave–current interactions in stratified flows with piecewise-constant vorticity was provided in the recent works [12,13,31].

Our approach is based on the *spatial dynamics* method, where we formulate the problem as a *spatial reversible dynamical system*

$$\frac{dU}{dx}=L_{\alpha_0,\rho_0}U+N(U;\varepsilon),\quad U(x)\in\mathbb{D},\tag{1}$$

where ε is a bifurcation parameter introduced by perturbing physical parameters α and ρ present in the problem ($\alpha=\frac{g\rho_1 d^3}{m_1^2}$ and $\rho=\frac{\rho_2}{\rho_1}$ with g the acceleration due to gravity and m_1 a constant related to mass flux) around the reference values α_0 and ρ_0, $\mathbb{D}$ is an appropriate infinite-dimensional Banach space consisting of functions of y, and where L_{α_0,ρ_0} is a linear operator depending on the pair (α_0,ρ_0) and $N(U,\varepsilon)$ is the nonlinear part in (U,ε) with $U=0$ a fixed point. This spatial dynamics method was already used to establish many existence theories for *irrotational* water wave problem, and also used by Groves and Wahlén [21,22]