Nonlinear Analysis 160 (2017) 25–43

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

On the well-posedness of various one-dimensional model equations for fluid motion

Hantaek Bae[a,*], Dongho Chae[b], Hisashi Okamoto[c]

[a] *Department of Mathematical Sciences, Ulsan National Institute of Science and Technology, Republic of Korea*
[b] *Department of Mathematics, Chung-Ang University, Republic of Korea*
[c] *Department of Mathematics, Gakushuin University, Tokyo, Japan*

ARTICLE INFO

Article history:
Received 5 December 2016
Accepted 6 May 2017
Communicated by Enzo Mitidieri

Keywords:
1D model equations
Nonlocal velocity
Local well-posedness
Blow-up criterion
Global well-posedness

ABSTRACT

We consider 1D equations with nonlocal velocity of the form

$$w_t + uw_x + \delta u_x w = -\nu \Lambda^\gamma w$$

where the nonlocal velocity u is given by (1) $u = (1-\partial_{xx})^{-\beta} w$, $\beta > 0$ or (2) $u = \mathcal{H}w$ ($\mathcal{H}$ is the Hilbert transform). In this paper, we address several local well-posedness results with blow-up criteria for smooth initial data. We then establish the global well-posedness by using the blow-up criteria.

1. Introduction

The homogeneous incompressible Euler equations are the following system of partial differential equations for (v, p) on $\mathbb{R}^n$:

$$\begin{aligned} &v_t + v \cdot \nabla v = -\nabla p, \\ &\mathrm{div} v = 0, \end{aligned} \tag{1.1}$$

where $v(t,x) = (v_1(t,x), v_2(t,x), \ldots, v_n(t,x))$ is the velocity field and $p(t,x)$ is the pressure. In three dimensions, we introduce the vorticity $\omega = \nabla \times v$, and write (1.1) in vorticity form

$$\omega_t + v \cdot \nabla \omega = \omega \cdot \nabla v, \tag{1.2a}$$

$$v = (-\Delta)^{-1} \nabla \times \omega. \tag{1.2b}$$

* Corresponding author.
E-mail addresses: hantaek@unist.ac.kr (H. Bae), dchae@cau.ac.kr (D. Chae), hisashi.okamoto@gakushuin.ac.jp (H. Okamoto).

http://dx.doi.org/10.1016/j.na.2017.05.002

It is well-known that the 3D Euler equations are locally well-posed with initial data in $H^s(\mathbb{R}^3)$, $s > \frac{5}{2}$. Moreover, the Beale–Kato–Majda criterion [3] tells that

$$\limsup_{t\nearrow T^*} \|v(t)\|_{H^s} = \infty \quad \text{if and only if} \quad \int_0^{T^*} \|\omega(s)\|_{L^\infty} ds = \infty.$$

However, the question of the global well-posedness or finite time blow-up of the 3D Euler equations is not answered yet.

To understand the nonlinear and nonlocal structure of the Euler equations, the 1D reduction was initiated by Constantin–Lax–Majda [12]. They proposed the following 1D model

$$w_t = w\mathcal{H}w, \quad \mathcal{H}w(x) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{w(y)}{x-y} dy \tag{1.3}$$

by replacing the Calderon–Zygmund type singular integral operator $\mathcal{P}$ in $\nabla v = \mathcal{P}\omega$ by the Hilbert transform $\mathcal{H}$ and by ignoring the convection term, and proved that $\mathcal{H}w$ blows up in finite time if

$$\{x \in \mathbb{R} : w_0(x) = 0, \quad \mathcal{H}w_0(x) > 0\} \neq \emptyset.$$

Motivated by this work, several nonlocal and quadratically nonlinear models were proposed and analyzed. In particular, De Gregorio [15] re-installed the convection term and the Biot–Savart law in the Euler equations and proposed the following equation as a model equation of (1.2)

$$w_t + vw_x - v_x w = 0, \tag{1.4a}$$

$$v_x = \mathcal{H}w. \tag{1.4b}$$

In this case, we still do not know whether smooth solutions of (1.4) blow up in finite time or not.

In this paper, we consider two types of 1D models from several physically important equations. We first consider the equations

$$\text{Model 1}: \begin{cases} w_t + uw_x + \delta u_x w = -\nu \Lambda^\gamma w \\ u = (1-\partial_{xx})^{-\beta} w \end{cases} \tag{1.5}$$

where $\delta \in \mathbb{R}$, $\nu, \gamma, \beta \geq 0$, and $\Lambda^\gamma = \sqrt{-\Delta}^\gamma$ is the fractional Laplacian defined in (2.1). We note that (1.5) represents many physical equations. For example, for $\beta = 0$ (1.5) is equivalent to the Burgers' equation with the fractional Laplacian, while for $\beta = 1$, $\nu = 0$ and $\delta = 2$ (1.5) becomes the Camassa–Holm equation [4]. (1.5) is also closely related to a generalized Proudman–Johnson equation [27–29]:

$$f_{txx} + ff_{xxx} + \delta f_x f_{xx} = \nu f_{xxxx} \tag{1.6}$$

which is derived from the 2D incompressible Navier–Stokes equations via the separation of variables. Taking $w = f_{xx}$, (1.6) is reduced to

$$w_t + fw_x + \delta f_x w = \nu w_{xx}, \quad w = f_{xx}.$$

In this paper, we also consider the following equations

$$\text{Model 2}: \begin{cases} w_t + uw_x + \delta u_x w = -\nu \Lambda^\gamma w \\ u = \mathcal{H}w \end{cases} \tag{1.7}$$

which are model equations of the 2D surface quasi-geostrophic equations.

Before proceeding to state our results, we review some related results.