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Liouville type theorems for stable solutions of p-Laplace equation in $\mathbb{R}^N$ [☆]

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ABSTRACT

In this paper we prove the Liouville type theorems for stable solution of the p-Laplace equations

$$-\Delta_p u = f(x)e^u, \quad \text{in } \mathbb{R}^N \tag{0.1}$$

and

$$\Delta_p u = f(x)u^{-q}, \quad \text{in } \mathbb{R}^N, \tag{0.2}$$

where $2 \le p < N, q > 0$ and the nonnegative function $f(x) \in L^1_{loc}(\mathbb{R}^N)$ such that $f(x) \ge c_0|x|^a$ as $|x| \ge R_0$ with $a > -p$ and some constants $R_0, c_0 > 0$. The results hold true for $2 \le N < \mu_0(p,a)$ in (0.1) and for $q > q_c(p,N,a)$ in (0.2). Here μ_0 and q_c are new exponents, which are always large than the classical critical ones and depend on the parameters p, a and N.

1. Introduction and main results

In this paper we study the Liouville type theorems for stable solutions to the following equations

$$-\Delta_p u = f(x)e^u, \quad \text{in } \mathbb{R}^N \tag{1.1}$$

and

$$\Delta_p u = f(x)u^{-q}, \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

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where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and $2 \le p < N$, $f(x) \in L^1_{loc}(\mathbb{R}^N)$ is nonnegative. The exact assumption on $f(x)$ will be given in (H_1) below.

Definition 1.1 (*see [20]*)**.** Let $\Omega \subset \mathbb{R}^N$ be a regular domain (bounded or not) and $g(x,\cdot): \mathbb{R} \to \mathbb{R}$ be a C^1 function for almost every $x \in \Omega$. We say that u is a weak solution of

$$-\Delta_p u = g(x,u), \quad \text{in } \Omega, \tag{1.3}$$

if $u \in C^{1,\omega}_{loc}(\Omega)(0 < \omega < 1)$ verifies $g(x,u) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\zeta dx = \int_\Omega g(x,u)\zeta dx, \quad \forall \zeta \in C^1_0(\Omega). \tag{1.4}$$

Let u be a weak solution of (1.3). We say that u is stable if $g_u(x,u) \in L^1_{loc}(\Omega)$ and

$$Q_u(\zeta) := \int_\Omega |\nabla u|^{p-2}|\nabla\zeta|^2 dx + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u\cdot\nabla\zeta)^2 dx - \int_\Omega g_u(x,u)\zeta^2 dx \ge 0, \tag{1.5}$$

for every $\zeta \in C^1_0(\Omega)$.

The Morse index of a solution u, $i(u)$ is defined as the maximal dimension of all subspace X of $C^1_0(\Omega)$ such that $Q_u(\zeta) < 0$ for any $\zeta \in X \setminus \{0\}$. Clearly, u is stable if and only if $i(u) = 0$.

Furthermore, a weak solution u of $-\Delta_p u = g(x,u)$ in Ω is said to be stable outside a compact set $\mathcal{K} \subset \Omega$ if $Q_u(\zeta) \ge 0$ for any $\zeta \in C^1_0(\Omega \setminus \mathcal{K})$. Recall that any finite Morse index solution u is stable outside a compact set $\mathcal{K} \subset \Omega$.

We note that the $C^{1,\omega}$ regularity assumption is natural to the solution of (1.3) duo to the results in [6,15,19].

Remark 1.1. If u is a stable weak solution of (1.1), then it follows from (1.5) that

$$\int_{\mathbb{R}^N} f(x)e^u\zeta^2 dx \le (p-1)\int_{\mathbb{R}^N} |\nabla u|^{p-2}|\nabla\zeta|^2 dx, \quad \forall\zeta \in C^1_0(\mathbb{R}^N). \tag{1.6}$$

Similarly, if u is a stable positive solution of (1.2), we have from (1.5) that

$$q\int_{\mathbb{R}^N} f(x)u^{-q-1}\zeta^2 dx \le (p-1)\int_{\mathbb{R}^N} |\nabla u|^{p-2}|\nabla\zeta|^2 dx, \quad \forall\zeta \in C^1_0(\mathbb{R}^N). \tag{1.7}$$

We recall that Liouville type theorem is the nonexistence of nontrivial solution in the entire space $\mathbb{R}^N$. The classical Liouville theorem stated that a bounded harmonic (or holomorphic) function defined in entire space $\mathbb{R}^N$ must be constant. This theorem, known as Liouville Theorem, was first announced in 1844 by Liouville [16] for the special case of a doubly-periodic function. Later in the same year, Cauchy [2] published the first proof of the above stated theorem. In 1981, Gidas and Spruck established in pioneering article [12] the optimal Liouville type result for nonnegative solutions of equation

$$-\Delta u = |x|^a u^q, \quad \text{in } \mathbb{R}^N. \tag{1.8}$$

They proved that (1.8) with $a = 0$ and $q > 1$ has no positive solution if and only if $1 < q < q_s = \frac{N+2}{N-2}(=\infty)$ if $N = 2$.

The case $a \ne 0$ is less understood. Let us introduce the Hardy–Sobolev exponent

$$q_s(a) = \frac{N+2+2a}{N-2}; \quad q_s(a) = \infty, \quad \text{if } N = 2. \tag{1.9}$$

In the class of radial solutions, the Liouville property was solved [1,18].