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Nonsmooth Morse–Sard theorems

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ABSTRACT

We prove that every function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies that the image of the set of critical points at which the function f has Taylor expansions of order $n-1$ and non-empty subdifferentials of order n is a Lebesgue-null set. As a by-product of our proof, for the proximal subdifferential ∂_P, we see that for every lower semicontinuous function $f : \mathbb{R}^2 \to \mathbb{R}$ the set $f(\{x \in \mathbb{R}^2 : 0 \in \partial_P f(x)\})$ is $\mathcal{L}^1$-null.

1. Introduction and main results

The main purpose of this paper is to provide nonsmooth versions of the Morse–Sard Theorem for real-valued functions defined on $\mathbb{R}^n$. Recall that the Morse–Sard theorem [26,32] states that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k, where $k = n - m + 1$, then the set of critical values of f has measure zero in $\mathbb{R}^m$. A famous example of Whitney's [36] shows that this classical result is sharp within the classes of functions C^j. Nevertheless several generalizations of the Morse–Sard theorem for other classes of functions have appeared in the literature; see [2,5,6,8,9,13,15,17,19–22,27–31,34,37] and the references therein. We cannot state all of the very interesting results of the rich literature concerning this topic; instead, because of its pointwise character which is closely related to our results, let us only mention that Bates proved in [5] that if $f \in C^{k-1,1}(\mathbb{R}^n, \mathbb{R}^m)$ (i.e., if $f \in C^{k-1}$ and $D^{k-1}f$ is Lipschitz) then the conclusion of the Morse–Sard theorem still holds true. In [3] we gave an abstract version of the Morse–Sard theorem which allows us to recover a previous result of De Pascale's for the class of Sobolev functions [13], as well as a refinement of Bates' theorem which only requires f to be $k-1$ times continuously differentiable and to satisfy a Stepanoff

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condition of order k, namely that

$$\limsup_{h\to 0} \frac{|f(x+h) - f(x) - Df(x)(h) - \cdots - \frac{1}{(k-1)!}D^{k-1}f(x)(h^{k-1})|}{|h|^k} < \infty$$

for every $x \in \mathbb{R}^n$. As a referee of the present paper pointed out, this result can also be easily proved, and even generalized, by using some ideas of the proof of [24, Theorem 1]; see the Appendix.

In the present paper we will look at the case $m = 1$ more closely, and we will study the question as to what extent one-sided Taylor expansions (that is, viscosity subdifferentials) of order n are sufficient to ensure that a given function $f : \mathbb{R}^n \to \mathbb{R}$ has the Morse–Sard property. The results that we obtain generalize many of the previous versions of Morse–Sard Theorem and do not require that the function f be C^{n-1} smooth (nor even two times differentiable). They are meant to complement the nonsmooth versions of the Morse–Sard theorem for subanalytic functions and for continuous selections of compactly indexed countable families of C^n functions on $\mathbb{R}^n$ that were established in [4,7].

For an integer $n \geq 2$, we will say that a function $f : \mathbb{R}^N \to \mathbb{R}^m$ has a Taylor expansion of order $n-1$ at x provided there exist k-homogeneous polynomials P_x^k, $k = 1, \ldots, n-1$, such that

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - P_x^1(h) - P_x^2(h) - \cdots - P_x^{n-1}(h)}{|h|^{n-1}} = 0.$$

If there exist such polynomials then they are unique. Also note that if a function f has Taylor expansion of order $n-1$ at a point x, then it is differentiable at x and the differential $Df(x)$ equals the linear function P_x^1; however $D^j f(x)$ does not necessarily exist for $j \geq 2$. On the other hand, if f is $n-1$ times differentiable at x then f has a Taylor expansion of order $n-1$ at x, and $P_x^k = \frac{1}{k!}D^k f(x)$ for every $k = 1, \ldots, n-1$. For more information on Taylor expansions and its relation with approximate differentiability and Lusin properties of higher order, see [23,24].

Let us now explain what we mean by a subdifferential of order n. Probably, the most natural way to define a subdifferential $\tilde{\partial}^n f(x_0)$ of order n of a lower semicontinuous function $f : \mathbb{R}^N \to \mathbb{R}$ at a point x_0 is as the set of n-tuples $(P_1, \ldots, P_n) \in \mathcal{P}(^1\mathbb{R}^N) \times \cdots \mathcal{P}(^n\mathbb{R}^N)$ such that

$$\liminf_{x\to x_0} \frac{f(x) - f(x_0) - P_1(x-x_0) - \cdots - P_n(x-x_0)}{|x-x_0|^n} \geq 0.$$

Here $\mathcal{P}(^k\mathbb{R}^N)$ denotes the space of k-homogeneous polynomials on $\mathbb{R}^N$, which is endowed with the norm

$$\|P\| = \sup_{|v|=1} |P(v)|.$$

In the case $n = 2$ this definition agrees with the standard viscosity subdifferential of order 2; see [12] and the references therein. It is easy to see that if $(P_1, \ldots, P_n) \in \tilde{\partial}^n f(x_0)$ then $(P_1, \ldots, P_{n-1}) \in \tilde{\partial}^{n-1} f(x_0)$. It is also clear that if the polynomial $\varphi(x) = f(x_0) + P_1(x-x_0) + \cdots P_{n-1}(x-x_0)$ satisfies $\varphi \leq f$ on a neighborhood of x_0, then $(P_1, \ldots, P_{n-1}) \in \tilde{\partial}^{n-1} f(x_0)$. For n odd, the converse is partially true, in the following sense: if $(P_1, \ldots, P_{n-1}) \in \tilde{\partial}^{n-1} f(x_0)$ and $\varepsilon > 0$, then the polynomial $\varphi(x) = f(x_0) + P_1(x-x_0) + \cdots P_{n-1}(x-x_0) - \varepsilon|x-x_0|^{n-1}$ is less than or equal to f on a neighborhood of x_0 (this is not necessarily true if n is even). Hence, we have the following.

Proposition 1.1. *If $(\zeta, P) \in \tilde{\partial}^2 f(x_0)$ and $P_\varepsilon(h) = P(h) - \varepsilon|h|^2$ then $(\zeta, P_\varepsilon, 0, \ldots, 0) \in \tilde{\partial}^n f(x_0)$ for every $\varepsilon > 0$ and every $n \geq 3$.*

However, this does not imply that $(\zeta, P, 0) \in \tilde{\partial}^3 f(x_0)$. In particular, we see that the subdifferential $\tilde{\partial}^n f(x_0)$ as a subset of $\mathcal{P}(^1\mathbb{R}^N) \times \cdots \mathcal{P}(^n\mathbb{R}^N)$, is not necessarily closed for $n \geq 2$ (although it is always closed for $n = 1$).