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A critical problem on the Hardy–Sobolev inequality in boundary singularity case

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ABSTRACT

We study a Neumann problem with the Hardy–Sobolev nonlinearity. In boundary singularity case, the impact of the mean curvature at singularity on existence of least-energy solution is well known. Existence and nonexistence of least-energy solution is studied by Hashizume (2017) except for lower dimension case. In this paper, we improve this previous work. More precisely, we study four dimensional case and show existence of minimizer in critical case in some sense.

1. Introduction

Let $N \geq 3$, Ω be smooth bounded domain in $\mathbb{R}^N$, $0 < s < 2$, $2^*(s) = 2(N-s)/(N-2)$ and $\lambda > 0$. The bounded embedding from $H^1(\Omega)$ to $L^{2^*(s)}(\Omega, |x|^{-s}dx)$ leads to the Hardy–Sobolev inequality

$$\mu_{s,\lambda}^N(\Omega)\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{2/2^*(s)} \leq \int_\Omega (|\nabla u|^2 + \lambda u^2)dx,$$

where the constant $\mu_{s,\lambda}^N(\Omega)$ is the largest possible constant defined by

$$\mu_{s,\lambda}^N(\Omega) = \inf_{u \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^2 + \lambda u^2)dx}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{2/2^*(s)}}.$$

The Dirichlet case, that is, the minimization problem of

$$\mu_s^D(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{2/2^*(s)}}$$

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is studied by many researchers in both interior singularity case and boundary singularity case. In interior singularity case, properties of $\mu_s^D(\Omega)$ are similar to the best constant of the Sobolev inequality. More precisely, $\mu_s^D(\Omega)$ is independent of Ω and never achieved. In boundary singularity case, [6–8] showed that minimizer exists when the mean curvature of $\partial\Omega$ at 0 is negative. In addition, [6] prove the nonexistence result under the assumption $T(\Omega) \subset \mathbb{R}^N_+$ for some rotation T, where $\mathbb{R}^N_+$ is a half-space. After these works [11] investigated a generalized minimization problem concerning $\mu_s^D(\Omega)$.

Existence and nonexistence of minimizer of $\mu^N_{s,\lambda}(\Omega)$ have been studied by [6,9]. In [6], they showed the existence of minimizer under the positivity of the mean curvature at 0. However the nonpositive mean curvature case was not dealt with in [6]. Recently, some part of this problem have been clarified by [9]. The result of nonpositive mean curvature case is completely different from that of positive mean curvature case. Concerning existence of minimizer of $\mu^N_{s,\lambda}(\Omega)$ in nonpositive mean curvature case we obtained the following result;

Theorem 1.1 ([9]). *Assume that $N \geq 5$ and the mean curvature of $\partial\Omega$ at 0 is nonpositive. Then there exists a positive constant $\lambda_* = \lambda_*(\Omega)$ such that*

(i) *If $0 < \lambda < \lambda_*$ then $\mu^N_{s,\lambda}(\Omega)$ is attained.*
(ii) *If $\lambda > \lambda_*$ then $\mu^N_{s,\lambda}(\Omega)$ is not attained.*

This situation is closed to the three dimensional case of the minimization problem introduced by Brezis and Nirenberg [3]:

$$S_\lambda(\Omega) := \inf_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^2 + \lambda u^2)dx}{\left(\int_\Omega |u|^{\frac{2N}{N-2}} dx\right)^{N/N-2}}, \quad \lambda \in \mathbb{R}.$$

In [3] they proved that there exists $\tilde{\lambda}_*(\Omega) < 0$ such that $S_\lambda(\Omega)$ is attained when $\lambda < \tilde{\lambda}_*(\Omega)$ and $S_\lambda(\Omega)$ is not attained when $\lambda > \tilde{\lambda}_*(\Omega)$. In addition, it is also proved that if Ω is a ball, existence of a minimizer is equivalent to $\lambda < \lambda_*(\Omega)$. After that, by [5] this result was extended to the general bounded domain case.

Our main purpose of this paper is to improve Theorem 1.1. More precisely, we investigate the case when $N = 4$ and the case when $\lambda = \lambda_*$.

What is related to the minimization problem $\mu^N_{s,\lambda}(\Omega)$ is the following elliptic equation:

$$\begin{cases} -\Delta u + \lambda u = \dfrac{u^{2^*(s)-1}}{|x|^s}, \quad u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Least-energy solution of (1) is defined by solution of (1) attaining $\mu^N_{s,\lambda}(\Omega)$ and thus existence of least-energy solution of (1) is equivalent to existence of minimizer of $\mu^N_{s,\lambda}(\Omega)$.

Asymptotic behavior of least-energy solution of (1) and $\mu^N_{s,\lambda}(\Omega)$ as $\lambda \to \infty$ have been studied in [9]. These studies are natural because a least-energy solution exists for any λ when the mean curvature at 0 is positive as we know. However, not only that, these studies play a crucial role in studying the minimization problem $\mu^N_{s,\lambda}(\Omega)$. Theorem 1.1 asserts that least-energy solution of (1) does not exist for sufficiently large λ when the mean curvature at 0 is nonpositive. In order to prove this fact, we need to investigate the asymptotic behavior of least-energy solution and $\mu^N_{s,\lambda}(\Omega)$ as $\lambda \to \infty$ under the assumption of existence of least-energy solution for any λ. This technique of the asymptotic analysis in [9] is used everywhere in this paper.

Our main results is as follows:

Theorem 1.2. *Assume $N = 4$ and the mean curvature of $\partial\Omega$ at 0 is nonpositive. Then there exists a positive constant $\lambda_* = \lambda_*(\Omega)$ such that*