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The Evans–Krylov theorem for nonlocal parabolic fully nonlinear equations

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ABSTRACT

In this paper, we prove the Evans–Krylov theorem for nonlocal parabolic fully nonlinear equations.

1. Introduction

Evans and Krylov proved independently an interior regularity for elliptic partial differential equations which states that any solution $u \in C^2(B_1)$ of a uniformly elliptic and fully nonlinear concave equation $F(D^2u) = 0$ in the unit ball $B_1 \subset \mathbb{R}^n$ satisfies an interior estimate $\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\,\|u\|_{C^{1,1}(B_1)}$ with some universal constants $C > 0$ and $\alpha \in (0,1)$, so-called the *Evans–Krylov* theorem (see [6,11] and [3]). Recently, Caffarelli and Silvestre [4] proved a nonlocal elliptic version of the Evans–Krylov theorem which describes that any viscosity solution $u \in L^\infty(\mathbb{R}^n)$ of concave homogeneous equation on $B_1 \subset \mathbb{R}^n$ formulated by elliptic integro-differential operators of order $\sigma \in (0,2)$ satisfies an estimate $\|u\|_{C^{\sigma+\alpha}(B_{1/2})} \leq C\,\|u\|_{L^\infty(\mathbb{R}^n)}$ with universal constants $C > 0$ and $\alpha \in (0,1)$. This nonlocal result makes it possible to recover the Evans–Krylov

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theorem as $\sigma \to 2^-$. In this paper, we prove a parabolic version of the nonlocal elliptic result of Caffarelli and Silvestre.

We consider the linear *parabolic integro-differential operators* given by

$$\mathrm{L}u(x,t) - \partial_t u(x,t) = \text{p.v.} \int_{\mathbb{R}^n} \mu_t(u,x,y) K(y)\, dy - \partial_t u(x,t) \tag{1.1}$$

for $\mu_t(u,x,y) = u(x+y,t) + u(x-y,t) - 2u(x,t)$. Here we write $\mu(u,x,y) = u(x+y) + u(x-y) - 2u(x)$ if u is independent of t. We refer the detailed definitions of notations to [4,7–9]. Then we see that $\mathrm{L}u(x,t)$ is well-defined provided that $u \in \mathrm{C}^{1,1}_x(x,t) \cap \mathrm{B}(\mathbb{R}^n_T)$ where $\mathrm{B}(\mathbb{R}^n_T)$ denotes *the family of all real-valued bounded functions defined on* $\mathbb{R}^n_T := \mathbb{R}^n \times (-T, 0]$ and $\mathrm{C}^{1,1}_x(x,t)$ means $\mathrm{C}^{1,1}$-function in x-variable at a given point (x,t). Moreover, $\mathrm{L}u(x,t)$ is well-defined even for $u \in \mathrm{C}^{1,1}_x(x,t) \cap L^\infty_T(L^1_\omega)$ (see [10]).

We say that the operator L belongs to $\mathfrak{L}_0 = \mathfrak{L}_0(\sigma)$ if its corresponding kernel $K \in \mathcal{K}_0 = \mathcal{K}_0(\sigma)$ satisfies the uniform ellipticity assumption:

$$(2-\sigma)\frac{\lambda}{|y|^{n+\sigma}} \le K(y) \le (2-\sigma)\frac{\Lambda}{|y|^{n+\sigma}}, \quad 0 < \sigma < 2. \tag{1.2}$$

If $K(y) = c_{n,\sigma}|y|^{-n-\sigma}$ where $c_{n,\sigma} > 0$ is the normalization constant comparable to $\sigma(2-\sigma)$ given by

$$c_{n,\sigma} = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(y_1)}{|y|^{n+\sigma}}\, dy \right)^{-1},$$

then the corresponding operator is $\mathrm{L} = -(-\Delta)^{\sigma/2}$. Also we say the operator $\mathrm{L} \in \mathfrak{L}_0$ belongs to $\mathfrak{L}_1 = \mathfrak{L}_1(\sigma)$ if its corresponding kernel $K \in \mathcal{K}_1 = \mathcal{K}_1(\sigma)$ satisfies $K \in \mathrm{C}^1$ away from the origin and satisfies

$$|\nabla K(y)| \le \frac{C}{|y|^{n+1+\sigma}}. \tag{1.3}$$

Finally we say that the operator $\mathrm{L} \in \mathfrak{L}_1$ belongs to $\mathfrak{L}_2 = \mathfrak{L}_2(\sigma)$ if its corresponding kernel $K \in \mathcal{K}_2 = \mathcal{K}_2(\sigma)$ satisfies $K \in \mathrm{C}^2$ away from the origin and satisfies

$$|D^2 K(y)| \le \frac{C}{|y|^{n+2+\sigma}}. \tag{1.4}$$

The maximal operators are defined by

$$\begin{aligned} \mathbf{M}^+_0 u(x,t) &= \sup_{\mathrm{L}\in\mathfrak{L}_0} \mathrm{L}u(x,t) = (2-\sigma)\int_{\mathbb{R}^n} \frac{\Lambda\mu_t^+(u,x,y) - \lambda\mu_t^-(u,x,y)}{|y|^{n+\sigma}} dy, \\ \mathbf{M}^+_1 u(x,t) &= \sup_{\mathrm{L}\in\mathfrak{L}_1} \mathrm{L}u(x,t) \quad \text{and} \quad \mathbf{M}^+_2 u(x,t) = \sup_{\mathrm{L}\in\mathfrak{L}_2} \mathrm{L}u(x,t). \end{aligned}$$

We shall consider nonlinear integro-differential operators, which originates from stochastic control theory with jump processes related with

$$\mathbf{I}u(x,t) = \inf_{\beta\in\mathcal{B}} \mathrm{L}_\beta u(x,t),$$

where $\mathrm{L}_\beta u(x,t) = \text{p.v.} \int_{\mathbb{R}^n} \mu_t(u,x,y) K_\beta(y)\, dy$ (see [1,4,7,8,12,13] for the elliptic case and [9,10] for the parabolic case). In this paper, we are mainly interested in the nonlocal parabolic concave equations

$$\mathbf{I}u(x,t) - \partial_t u(x,t) = 0 \quad \text{in } Q_1. \tag{1.5}$$

[**Notations and Definitions**] Let $\sigma \in (0,2)$ and $r > 0$.

- Denote $Q_r = B_r \times I^\sigma_r$ and $Q_r(x,t) = Q_r + (x,t)$ for $(x,t) \in \mathbb{R}^n_T$, where $B_r(x)$ is the open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$, $B_r = B_r(0)$ and $I^\sigma_r = (-r^\sigma, 0]$.