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Poincaré inequalities for Littlewood–Paley operators

Liguang Liu[a], Dachun Yang[b,*], Liying Zhang[b]

[a] *Department of Mathematics, School of Information, Renmin University of China, Beijing 100872, China*
[b] *School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China*

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ABSTRACT

In this paper, the authors prove that the inequalities of Poincaré-type are preserved under the action of the Littlewood–Paley operators. Applications to boundedness of the Littlewood–Paley operators on Lipschitz spaces and Hajłasz–Sobolev spaces are considered.

1. Introduction

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class on the Euclidean space $\mathbb{R}^n$ equipped with the well-known classical topology, and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz distribution equipped with the weak-$*$ topology. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\ \widehat{\varphi} \subset \left\{x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2\right\}. \tag{1.1}$$

Here and hereafter, $\widehat{\varphi}$ denotes the *Fourier transform* of φ, namely, for any $\xi \in \mathbb{R}^n$,

$$\widehat{\varphi}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

* Corresponding author.
E-mail addresses: liuliguang@ruc.edu.cn (L. Liu), dcyang@bnu.edu.cn (D. Yang), zhangliying@mail.bnu.edu.cn (L. Zhang).

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For any $t \in (0, \infty)$ and $x \in \mathbb{R}^n$, let $\varphi_t(x) := t^{-n}\varphi(x/t)$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *Littlewood–Paley g-function* $g(f)$ is defined by setting

$$g(f)(x) := \left\{ \int_0^\infty |\varphi_t * f(x)|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \tag{1.2}$$

and the *Lusin-area integral S-function* $S(f)$ by setting

$$S(f)(x) := \left\{ \int_0^\infty \int_{|y-x|<t} |\varphi_t * f(y)|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{\frac{1}{2}}, \tag{1.3}$$

as well as the *Littlewood–Paley g_λ^*-function* $g_\lambda^*(f)$ by setting

$$g_\lambda^*(f)(x) := \left\{ \iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |\varphi_t * f(y)|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{\frac{1}{2}}, \tag{1.4}$$

where $\lambda \in (0, \infty)$ is a fixed parameter.

Throughout the paper, we use the following notation: for any locally integrable function f on $\mathbb{R}^n$ and for any ball $B \subset \mathbb{R}^n$, let

$$f_B := \frac{1}{|B|} \int_B f(x)\,dx. \tag{1.5}$$

The main result of this paper is as follows.

Theorem 1.1. *Let $p_0 \in [1, \infty)$. Assume that $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and, for any ball $B \subset \mathbb{R}^n$ of radius $r \in (0, \infty)$,*

$$\frac{1}{|B|} \int_B |f(x) - f_B|\,dx \le r^\alpha \left[\frac{\mu(B)}{|B|} \right]^{1/p_0}, \tag{1.6}$$

where either

(i) *$\alpha = 0$ and μ equals to the Lebesgue measure on $\mathbb{R}^n$, or*
(ii) *$\alpha \in (0, \min\{1, n/p_0\})$ and μ is a locally finite positive Borel measure.*

Assume that $p = 1$ when $\alpha = 0$, or $p \in [1, np_0/(n - \alpha p_0))$ when $\alpha \in (0, 1)$. Let T be the Littlewood–Paley g-function, or the Lusin-area S-function, or the Littlewood–Paley g_λ^*-function with $\lambda \in (2, \infty)$, respectively, as in (1.2)–(1.4). Then $T(f)$ is either infinite everywhere or finite almost everywhere and, in the latter case, for any ball $B \subset \mathbb{R}^n$,

$$\left[\frac{1}{|B|} \int_B |T(f)(x) - (T(f))_B|^p\,dx \right]^{1/p} \le C r^\alpha \left[\sup_{k \in \mathbb{N}} \frac{\mu(2^k B)}{|2^k B|} \right]^{1/p_0}, \tag{1.7}$$

where $(T(f))_B$ is as in (1.5) with f replaced by $T(f)$, and C is a positive constant independent of f and B.

It should be pointed out that, when $\alpha = 0$ and μ is the Lebesgue measure on $\mathbb{R}^n$, then the results of Theorem 1.1 directly implies the boundedness of the Littlewood–Paley operators on the space $\mathrm{BMO}(\mathbb{R}^n)$, which was proved by Meng and Yang [15]. Recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ is defined to be the set of all locally integrable functions f such that

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B|\,dx < \infty,$$

where f_B is as in (1.5) and the supremum is taken over all balls $B \subset \mathbb{R}^n$. Boundedness of the Littlewood–Paley operators and the Hardy–Littlewood maximal operator on $\mathrm{BMO}(\mathbb{R}^n)$-type spaces has been studied in