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Locally optimal configurations for the two-phase torsion problem in the ball

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ABSTRACT

We consider the unit ball $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ filled with two materials with different conductivities. We perform shape derivatives up to the second order to find out precise information about locally optimal configurations with respect to the torsional rigidity functional. In particular we analyse the role played by the configuration obtained by putting a smaller concentric ball inside Ω. In this case the stress function admits an explicit form which is radially symmetric: this allows us to compute the sign of the second order shape derivative of the torsional rigidity functional with the aid of spherical harmonics. Depending on the ratio of the conductivities a symmetry breaking phenomenon occurs.

1. Introduction

We will start by considering the following two-phase problem. Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be the unit open ball centred at the origin. Fix $m \in (0, \mathrm{Vol}(\Omega))$, where here we denote the N-dimensional Lebesgue measure of a set by $\mathrm{Vol}(\cdot)$. Let $\omega \subset\subset \Omega$ be a sufficiently regular open set such that $\mathrm{Vol}(\omega) = m$. Fix two positive constants σ_-, σ_+ and consider the following *distribution of conductivities*:

$$\sigma := \sigma_\omega := \sigma_- \mathbb{1}_\omega + \sigma_+ \mathbb{1}_{\Omega \setminus \overline{\omega}},$$

where by $\mathbb{1}_A$ we denote the characteristic function of a set A (i.e. $\mathbb{1}_A(x) = 1$ if $x \in A$ and vanishes otherwise) (see Fig. 1).

Consider the following boundary value problem:

$$\begin{cases} -\mathrm{div}\,(\sigma_\omega \nabla u) = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

We recall the weak formulation of (1.1):

$$\int_\Omega \sigma_\omega \nabla u \cdot \nabla \varphi = \int_\Omega \varphi \quad \text{for all } \varphi \in H_0^1(\Omega). \tag{1.2}$$

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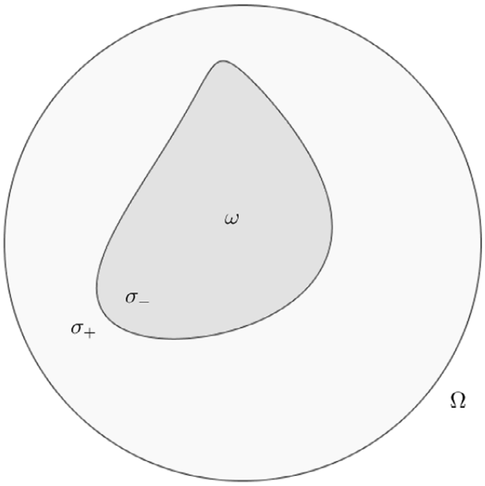

Fig. 1. Our problem setting.

Moreover, since σ_ω is piecewise constant, we can rewrite (1.1) as follows

$$\begin{cases} -\sigma_\omega \Delta u = 1 & \text{in} \quad \omega \cup (\Omega \setminus \overline{\omega}), \\ \sigma_- \partial_n u_- = \sigma_+ \partial_n u_+ & \text{on } \partial\omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where the following notation is used: the symbol $\mathbf{n}$ is reserved for the outward unit normal and $\partial_n := \frac{\partial}{\partial \mathbf{n}}$ denotes the usual normal derivative. Throughout the paper we will use the $+$ and $-$ subscripts to denote quantities in the two different phases (under this convention we have $(\sigma_\omega)_+ = \sigma_+$ and $(\sigma_\omega)_- = \sigma_-$ and our notations are "consistent" at least in this respect). The second equality of (1.3) has to be intended in the sense of traces. In the sequel, the notation $[f] := f_+ - f_-$ will be used to denote the jump of a function f through the interface $\partial\omega$ (for example, following this convention, the second equality in (1.3) reads "$[\sigma \partial_n u] = 0$ on $\partial\omega$").

We consider the following *torsional rigidity functional*:

$$E(\omega) := \int_\Omega \sigma_\omega |\nabla u_\omega|^2 = \int_\omega \sigma_- |\nabla u_\omega|^2 + \int_{\Omega \setminus \bar{\omega}} \sigma_+ |\nabla u_\omega|^2 = \int_\Omega u_\omega,$$

where u_ω is the unique (weak) solution of (1.1).

Physically speaking, we imagine our ball Ω being filled up with two different materials and the constants $\sigma_\pm$ represent how "hard" they are. The second equality of (1.3), which can be obtained by integrating by parts after splitting the integrals in (1.2), is usually referred to as *transmission condition* in the literature and can be interpreted as the continuity of the flux through the interface $\partial\omega$.

The functional E, then, represents the torsional rigidity of an infinitely long composite beam. Our aim is to study (locally) optimal shapes of ω with respect to the functional E under the fixed volume constraint. The one-phase version of this problem was first studied by Pólya. Let $D \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded Lipschitz domain. Define the following shape functional

$$\mathcal{E}(D) := \int_D |\nabla u_D|^2,$$

where the function u_D (usually called *stress function*) is the unique solution to

$$\begin{cases} -\Delta u = 1 & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases} \tag{1.4}$$