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Growth properties for Riesz potentials of functions in weighted variable $L^{p(\cdot)}$ spaces

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ABSTRACT

We study growth properties of spherical means for Riesz potentials of functions in weighted Lebesgue spaces of variable exponent. We also deal with Green potentials and monotone Sobolev functions.

1. Introduction

Let $\mathbf{R}^N$, $N \geq 2$, denote the N-dimensional Euclidean space. We use the notation $B(x,r)$ to denote the open ball centered at x with radius $r > 0$, whose boundary is denoted by $S(x,r)$. The L^q mean over the spherical surface $S(0,r)$ for a measurable function u is defined by

$$S_q(u,r) = \left(\frac{1}{|S(0,r)|} \int_{S(0,r)} |u(x)|^q \, dS(x) \right)^{1/q}$$
$$= \left(\frac{1}{\omega_{N-1}} \int_{S(0,1)} |u(r\sigma)|^q \, dS(\sigma) \right)^{1/q}$$

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when $1 \le q < \infty$, where $|S(0,r)| = \omega_{N-1} r^{N-1}$ with ω_{N-1} denoting the area of the unit sphere and dS denotes the surface area measure on $S(0,1)$. Gardiner [5, Theorem 2] showed that

$$\liminf_{r \to 1-} (1-r)^{(N-1)(1-1/q)} S_q(u,r) = 0$$

when u is a Green potential in the unit ball $\mathbf{B} = B(0,1)$, $(N-3)/(N-1) < 1/q \le (N-2)/(N-1)$ and $q > 0$, as an extension of the result by Stoll [21] in the plane case. The first author gave versions of Gardiner's result for Riesz potentials in his paper [13, Section 5]. The first and the third authors [17] studied the existence of boundary limits for BLD (Beppo Levi and Deny) functions u on the unit ball $\mathbf{B}$ of $\mathbf{R}^N$ satisfying

$$\int_{\mathbf{B}} |\nabla u(x)|^p (1-|x|)^\gamma \, dx < \infty, \tag{1.1}$$

where ∇ denotes the gradient, $1 < p < \infty$ and $-1 < \gamma < p-1$. In fact, it was shown that

$$\liminf_{r \to 1-} (1-r)^{(N-p+\gamma)/p-(N-1)/q} S_q(u,r) = 0$$

when $q > 0$ and $(N-p-1)/(p(N-1)) < 1/q < (N-p+\gamma)/(p(N-1))$. In [17], we also studied the existence of boundary limits for monotone BLD functions u on the unit ball $\mathbf{B}$ of $\mathbf{R}^N$ satisfying (1.1).

For $0 < \alpha < N$ and $f \in L^1_{loc}(\mathbf{B})$, we define the Riesz potential of order α by

$$I_\alpha f(x) = \int_{\mathbf{B}} |x-y|^{\alpha-N} f(y) \, dy.$$

We deal with functions $f \in L^1_{loc}(\mathbf{B})$ satisfying

$$\|f\|_{M^{p(\cdot),\omega}(\mathbf{B})} = \sup_{0<r<1} \omega(1-r) \|f\|_{L^{p(\cdot)}(\mathbf{B} \setminus B(0,r))} < \infty$$

with a variable exponent $p(\cdot)$ and a doubling weight ω; the space $M^{p(\cdot),\omega}(\mathbf{B})$ consisting of such functions is sometimes referred to as a (complementary) Morrey type space with variable exponent (see Section 2 for the definitions of $p(\cdot)$ and ω). For these spaces, we refer to [1–3] and [18]. Our main aim in this paper is to discuss the weighted limit :

$$\liminf_{r \to 1-} (1-r)^d \omega(1-r)^p S_q\left((I_\alpha f)^{p(r)}, r \right)$$

when $(1-|y|)^\kappa f(y) \in M^{p(\cdot),\omega}(\mathbf{B})$ for some $\kappa \ge 0$ (see Theorems 3.1 and 4.2 below); the exponent d will be given later.

Let $G(x,y)$ be the Green kernel on $\mathbf{B}$. We define the Green potential for $f \in L^1_{loc}(\mathbf{B})$ by

$$Gf(x) = \int_{\mathbf{B}} G(x,y) f(y) \, dy.$$

In Section 5, we study the existence of weighted spherical limits for Green potentials Gf with $(1-|y|)f(y) \in M^{p(\cdot),\omega}(\mathbf{B})$ as an extension of Gardiner [5, Theorem 2] (see Theorem 5.4 below).

A continuous function u on an open set Ω is called monotone in the sense of Lebesgue [8] if for every relatively compact open set $G \subset \Omega$,

$$\max_{\overline{G}} u = \max_{\partial G} u \quad \text{and} \quad \min_{\overline{G}} u = \min_{\partial G} u.$$

Harmonic functions on Ω are monotone in Ω. More generally, solutions of elliptic partial differential equations of second order and weak solutions for variational problems may be monotone (see [6]). See also [7,10,11,14,15,24,25] and [26]. The last section is concerned with weighted spherical limits for monotone Sobolev functions u with $|\nabla u(y)|^{p_1} \in M_A^{p(\cdot),\omega}(\mathbf{B})$ with $p_1 > N-1$ (see Theorem 6.1 below). See Section 6 for the definition of $M_A^{p(\cdot),\omega}(\mathbf{B})$. Essential tool in treating monotone functions is Lemma 6.4 below.

For related results on spherical means, see [12,13], [15,20,22] and [23]. We also refer the reader to the papers [9] and [19] for weighted integral means over balls.