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Modulation spaces and non-linear Hartree type equations

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ABSTRACT

We study the Cauchy problem for Hartree equation with cubic convolution nonlinearity $F(u) = (k \star |u|^2)u$ under a specified condition on potential k with Cauchy data in modulation spaces $M^{p,q}(\mathbb{R}^n)$. We establish global well-posedness results in $M^{p,p}(\mathbb{R}^n)$ with $1 \leq p < \frac{2n}{n+\nu}$, when $k(x) = \frac{\lambda}{|x|^\nu}$ ($\lambda \in \mathbb{R}$, $0 < \nu < min\{2, \frac{n}{2}\}$); in $M^{p,q}(\mathbb{R}^n)$ with $1 \leq q \leq p \leq 2$, when $k \in M^{\infty,1}(\mathbb{R}^n)$.

1. Introduction

The theory of modulation spaces has been developed substantially in the last decade. Modulation spaces provide quantitative information about time–frequency concentration of functions and distributions. The modulation spaces are defined in terms of short-time Fourier transform. The short-time Fourier transform of a function is defined as inner product of the function with respect to a time–frequency shift of another function, known as a window function (for precise definition see Section 2). Modulation spaces have found their usefulness in applications as well as in pure mathematics. The family of modulation spaces had been studied in time–frequency analysis since the 1980s and later was also used in pseudo-differential operators. They play a useful role in the theory of pseudo-differential operators [22].

Inspired from the work of Chadam–Glassey [7], in 1980s Ginibre–Velo [12] have studied the Schrödinger equation with cubic convolution nonlinearity due to both their strong physical background and theoretical importance (for instance, it appears in quantum theory of boson stars, atomic and nuclear physics, describing superfluids, etc.). Nonlinear Schrödinger equations (NLS) are important both for many different applications as well as a source of rich mathematical theory with several hard challenges still open. The NLS in the

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most common meaning contains a local nonlinearity given by a power of the local density, in particular the (de)focusing "cubic" NLS which arises e.g., in nonlinear optics or for Bose Einstein condensates. A class of NLS with a "non-local" nonlinearity that we call Hartree type occur in the modelling of quantum semiconductor devices. In this paper, we consider the existence of global solutions to the Cauchy problem for the Hartree type equation:

$$iu_t + \Delta u = (k \star |u|^2)u, \quad u(x, t_0) = u_0(x); \tag{1.1}$$

where $u(x,t)$ is a complex valued function on $\mathbb{R}^n \times \mathbb{R}$, Δ is the Laplacian in $\mathbb{R}^n$, u_0 is a complex valued function on $\mathbb{R}^n$, k is some suitable potential (function) on $\mathbb{R}^n$, time $t_0 \in \mathbb{R}$, and $\star$ denotes the convolution in $\mathbb{R}^n$.

In subsequent years the local and global well-posedness, regularity, and scattering theory for Eq. (1.1) have attracted a lot of attention by many mathematicians. Almost exclusively, the techniques developed so far restrict to Cauchy problems with initial data in a Sobolev space, mainly because of the crucial role played by the Fourier transform in the analysis of partial differential operators, see [5,6,12]. In particular, in 1998s Hayashi and Naumkin [16] have studied the Cauchy problem (1.1) with Hartree potential in the space dimensions $n \geq 2$ under the conditions that the initial data

$$u_0 \in H^{\nu,0} \cap H^{0,\nu}, \text{ with } \nu > n/2$$

and the norm $\|u_0\|_{\nu,0} + \|u_0\|_{0,\nu}$ is sufficiently small, where $H^{\mu,\nu}$ is the usual weighted Sobolev space defined by

$$H^{\mu,\nu} = \{f \in L^2; \|f\|_{\mu,\nu} = \|(1+|x|^2)^{\mu/2} (I-\Delta)^{\nu/2}\|_{L^2} < \infty\}, \ \mu, \nu \in \mathbb{R}.$$

We see that over the past ten years there has been increasing interest for many mathematicians to consider Cauchy data in modulation spaces $M^{p,q}(\mathbb{R}^n)$ (Definition 2.4) for nonlinear dispersive equations because these spaces are rougher than any given one in a fractional Bessel potential space and this low-regularity is desirable in many situations. For instance, we mention the local well-posedness result of Schrödinger equation, especially with power type nonlinearity $F(u) = |u|^{2k}u$ $(k \in N)$ is obtained in [3,8,24] with Cauchy data from $M^{p,1}(\mathbb{R}^n)$ and a global existence result in [15,23] with small initial data from $M^{p,1}(\mathbb{R}^n)(1 \leq p \leq 2)$, see also [9,18]. However, the global well-posedness result for the large initial data (without any restriction to initial data) in modulation space is not yet clear, in fact it is an open question [19, p. 280], because one of the main obstacle is a lack of useful conservation laws in modulation spaces by which one can guarantee the global existence result. In this direction, recently, D. G. Bhimani [4] has shown the global well-posedness result to the Cauchy problem for the Hartree type equations with $u_0 \in M^{1,1}$.

In 1978s, Lin and Strauss [17] construct a complete theory of scattering for the nonlinear Schrödinger (NLS) equation

$$iu_t + (1/2)\Delta u = f(u)$$

in space dimension $n = 3$ with nonlinear interaction $f(u) = |u|^{p-1}u$ and $8/3 < p < 5$. Parallel and sub-sequent developments included the construction of a complete theory of scattering in the space

$$\Sigma = H^1 \cap \mathcal{F}H^1,$$

where H^1 is the usual Sobolev space and $\mathcal{F}$ the Fourier transform, in arbitrary space dimension, both for the NLS equation and for the Hartree equation. Asymptotic completeness is proved there by the use of the approximate conservation law associated with the approximate pseudo-conformal invariance of the NLS and Hartree equations. The class of interactions thereby covered includes the potential $k(x) = \frac{C}{|x|^\nu}$ with $C > 0$