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Abstract Cauchy problems for quasilinear operators whose domains are not necessarily dense or constant

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ABSTRACT

The solvability of the abstract Cauchy problem for the quasilinear evolution equation $u'(t) = A(u(t))u(t)$ for $t > 0$ and $u(0) = u_0 \in D$ is discussed. Here $\{A(w); w \in Y\}$ is a family of closed linear operators in a real Banach space X such that $Y \subset D(A(w)) \subset \overline{Y}$ for $w \in Y$, Y is another Banach space which is continuously embedded in X, and D is a closed subset of Y. The existence and uniqueness of C^1 solutions to the Cauchy problem is proved without assuming that Y is dense in X or $D(A(w))$ is independent of w. The abstract result is applied to obtain an L^1-valued C^1-solution to a size-structured population model.

1. Introduction

The 'global' well-posedness of the abstract Cauchy problem for the quasi-linear evolution equation

$$(\mathrm{QE}; u_0) \quad \begin{cases} u'(t) = A(u(t))u(t) & \text{for } t \in [0, \tau), \\ u(0) = u_0 \end{cases}$$

is discussed, where $\{A(w); w \in Y\}$ is a family of closed linear operators in a real Banach space X such that $D(A(w)) \supset Y$ for $w \in Y$, and Y is another Banach space which is continuously embedded in X.

The study of 'local' well-posedness of the Cauchy problem (QE;u_0) was initiated by Kato [9] in the case where X and Y are reflexive and Y is dense in X. After his pioneering work, Sanekata [18] successfully eliminated the reflexivity condition and gave an application to quasilinear hyperbolic systems in spaces of continuous functions (see also Kato [10,11]).

In getting solutions to some second order equation with Wentzell boundary condition in the space of continuous functions, the domains of associated differential operators are not generally dense in the

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underlying spaces. In order to study such problems in a unified and systematic way, Bátkai and Engel [5] discussed the second order abstract Cauchy problems for operators with Wentzell-type boundary conditions, using the theory of integrated semigroups developed by Arendt [3]. Their work motivated us [14] to investigate the 'global' well-posedness of the Cauchy problem (QE;u_0) in the case where $D(A(w)) = Y$ for $w \in Y$ and Y is not necessarily dense in X. In this connection, it should be noted that Da Prato and Sinestrari [6] first gave some interesting results on the inhomogeneous abstract Cauchy problem for a closed linear operator A in X satisfying the Hille–Yosida condition with the exception of the density of the domain of A. Their results have been extended to various types of equations by several authors (see [4,7,12,15–17,19,20]).

The purpose is to establish a 'global' well-posedness theorem (Theorem 2.7) of the abstract quasilinear evolution equation (QE) in the case where $D(A(w))$ is not in general equal to Y and Y is not always dense in X. This situation occurs in studying the well-posedness of a quasilinear size-structured model in the space of integrable functions (see Section 5). Since $D(A(w))$ is not assumed to equal Y, we employ a family $\{S(w); w \in Y\}$ of isomorphisms of Y onto X, according to the idea due to Kato [8]. Such a family is introduced to impose a commutator condition on $A(w)$ and $S(w)$, which plays an important role in proving that a family of approximate solutions converges in Y. However, the commutator condition in the sense of Kato [8] is not in general satisfied in a quasilinear size-structured model discussed in [1,2]. To overcome such difficulty, we use a projection P in $B(X)$ with $P(X) = \overline{Y}$ such that

$$S(w)PA(w)S(w)^{-1} = A(w)P + B(w) \quad \text{for } w \in Y,$$

where $\{B(w); w \in Y\}$ is a family in $B(X)$ satisfying appropriate conditions. This is the main feature of this paper. In the final section, the abstract result is applied to obtain an L^1-valued C^1-solution to a size-structured population model with initial data given in $W^{1,1}$ under a compatibility condition.

2. Assumptions and the main theorem

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces and Y is assumed to be continuously embedded in X. The norm closure of Y in X is denoted by $\overline{Y}$. The symbol $B(X,Y)$ stands for the Banach space of all bounded linear operators on X to Y with usual operator norm $\|\cdot\|_{X,Y}$. The norm of $B(X,X)$ is denoted simply by $\|\cdot\|_X$. The notations $a \wedge b := \min(a,b)$, $a \vee b := \max(a,b)$, $\mathbb{R}_+ := [0,\infty)$ and $B_Y(r) := \{w \in Y; \|w\|_Y \le r\}$ are used.

Let $\{A(w); w \in Y\}$ be a family of closed linear operators in X such that

$$Y \subset D(A(w)) \subset \overline{Y} \quad \text{for } w \in Y,$$

and the following assumptions are imposed:

(A1) For each $r > 0$ there exists $\omega_X(r) \ge 0$ such that $A(w) - \omega_X(r)I$ is m-dissipative in X for $w \in B_Y(r)$.
(A2) For each $r > 0$ there exists $L_A(r) > 0$ such that

$$\|A(w) - A(z)\|_{Y,X} \le L_A(r)\|w - z\|_X \text{ for } w, z \in B_Y(r).$$

(S) There exist a projection $P \in B(X)$, a family $\{S(w); w \in Y\}$ of isomorphisms of Y onto X and a family $\{B(w); w \in Y\}$ in $B(X)$ such that

$$S(w)PA(w)S(w)^{-1} = A(w)P + B(w) \tag{2.1}$$

for $w \in Y$, and the following conditions are satisfied: