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Vanishing theorems for f-harmonic forms on smooth metric measure spaces[☆]

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ABSTRACT

In this paper, we first establish a monotonicity formula for vector bundle-valued f-harmonic p-forms on a smooth metric measure space provided $\langle \nabla f, \nabla r\rangle$ is less that an explicit constant. As applications, we get some vanishing theorems for L^2 f-harmonic forms on concrete geometric models. In the second part, for a metric measure space with nonnegative ∞-Bakry–Émery–Ricci curvature and with moderate volume growth, we prove that any bounded f-harmonic 1-form must be parallel. Moreover, some vanishing theorems under nonnegative m-Bakry–Émery–Ricci curvature assumption are also proved. Finally, we consider smooth metric measure spaces with weighted Poincaré inequality and show some vanishing theorems for L^q f-harmonic 1-forms.

1. Introduction

A smooth metric measure space is a triple $(M, g, e^{-f}dv_g)$, where (M, g) is an n-dimensional Riemannian manifold, dv_g is the Riemannian volume element and f is a smooth positive function on M. Let $\xi : E \to M$ be a smooth Riemannian vector bundle over M with a metric compatible connection ∇^E. Set $A^p(\xi) = \Gamma(\Lambda^p T^*M \otimes E)$ the space of smooth p-forms on M with values in the vector bundle $\xi : E \to M$. The exterior covariant differentiation $d^\nabla : A^p(\xi) \to A^{p+1}(\xi)$ relative to the connection ∇^E is defined by (cf. [8])

$$(d^\nabla \omega)(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} (\nabla_{X_i}\omega)(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1}).$$

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The codifferential operator $\delta_f^\nabla : A^p(\xi) \to A^{p-1}(\xi)$ characterized as the adjoint of d^∇ with respect to the measure $e^{-f}dv$ is defined by $\delta_f^\nabla = \delta^\nabla + i_{\nabla f}$, that is,

$$(\delta_f^\nabla \omega)(X_1, \dots, X_{p-1}) = -\sum_i (\nabla_{e_i}\omega)(e_i, X_1, \dots, X_{p-1}) + \omega(\nabla f, X_1, \dots, X_{p-1}).$$

Thus, we can introduce a new notion of the Laplace–Beltrami operator $\triangle_f$ as

$$\triangle_f = d^\nabla \delta_f^\nabla + \delta_f^\nabla d^\nabla = \triangle + (d^\nabla i_{\nabla f} + i_{\nabla f} d^\nabla),$$

where $i_{\nabla f}$ denotes the interior product by ∇f. When acting on smooth functions on M, it is easy to see that $\triangle_f$ is self-adjoint in a weighted L^2 space, that is,

$$\int_M u \triangle_f v e^{-f} dv = -\int_M \langle \nabla u, \nabla v \rangle e^{-f} dv$$

for any $u, v \in C_0^\infty(M)$.

Denote Ric and Hess(f) to be the Ricci curvature and the Hessian of f respectively. Then the ∞-Bakry–Émery–Ricci curvature and the m-Bakry–Émery–Ricci curvature are defined by

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}(f) \quad \text{and} \quad \mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess}(f) - \frac{\nabla f \otimes \nabla f}{m-n},$$

respectively, where $m \geq n$ and $m = n$ if and only if f is constant. These curvature quantities are very useful in the study of smooth metric measure spaces, since they appear naturally in the generalized Bochner formula in the weighted setting. In this curvature point of view, gradient Ricci solitons, which defined by $\mathrm{Ric}_f = \lambda g$ for some function f and constant λ, can be seen as canonical examples of smooth metric measure spaces with the ∞-Bakry–Émery–Ricci curvature bounded below. The study of smooth metric measure spaces has been very active in recent years, partially due to the interest in the study of gradient Ricci solitons. For more interesting results about the geometry and analysis on general metric measure space, one can consult [13,16,17,23] and the references therein.

It is well-known that vanishing type theorems are important results in geometric analysis. Recently, there are some interesting vanishing type theorems on smooth metric measure spaces or gradient Ricci solitons. For example, in [16], Munteanu and Wang considered a smooth metric measure space with $\mathrm{Ric}_f \geq 0$. If the function f is of sublinear growth then any positive f-harmonic function on M must be constant. They also show that there does not exist any nontrivial sublinear growth f-harmonic function on M provided $\mathrm{Ric}_f \geq 0$ and the boundedness of f. In [20], Vieira proved a splitting type theorem for a smooth metric measure space with $\mathrm{Ric}_f \geq 0$ and $H^1(L^{2,f}(M)) \neq \{0\}$, where $H^p(L^{q,f}(M))$ denotes the space of all L^q f-harmonic p-forms on M. He also obtained that $H^1(L^{2,f}(M)) = \{0\}$ if the underlying manifold M satisfies $\mathrm{Ric}_f \geq 0$ and the bottom of the weighted spectrum $\lambda_1(\triangle_f) > 0$. For more vanishing results about harmonic forms or harmonic maps on smooth metric measure spaces or gradient Ricci solitons, we will refer the reader to [2,6,9,15,19,21–23] for recent progress on this topic and references therein.

The purpose of this paper is to study various vanishing theorems for f-harmonic p-forms on smooth metric measure spaces, under suitable geometric conditions. We first generalize the monotonicity formulae for vector bundle-valued harmonic p-forms on Riemannian manifolds to the case of smooth metric measure spaces, and then deduce some L^2 vanishing theorems.

Theorem 1.1. *Let $(M^n, g, e^{-f}dv)$, $n \geq 2$, be a complete smooth metric measure space and let $\xi : E \to M$ be a Riemannian vector bundle on M. Assume that (M, g) is a complete, noncompact Riemannian manifold with a pole o and its radial curvature satisfies one of the following three conditions:*

(i) $-\alpha^2 \leq K_r \leq -\beta^2$ *with* $\alpha > 0$, $\beta > 0$ *and* $\beta_p := n - 2p\frac{\alpha}{\beta} \geq 1$;

(ii) $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ *with* $\epsilon > 0$, $A \geq 0$, $0 \leq B < 2\epsilon$ *and* $\beta_p := n - (n-1)\frac{B}{2\epsilon} - 2pe^{A/2\epsilon} > 0$;