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Flux-limited solutions and state constraints for quasi-convex Hamilton–Jacobi equations in multidimensional domains

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ABSTRACT

Recently, Imbert and Monneau have introduced the so-called flux-limited formulation of Hamilton–Jacobi equation with state constraint boundary conditions. When the spatial domain is a bounded interval, they proved that the latter formulation is equivalent to the more classical one which was originally introduced by H–M. Soner. In the present paper, we aim to prove the same result for a multidimensional spatial domain. More precisely, we give the proof for a general bounded domain of $\mathbb{R}^d$ with a $\mathcal{C}^1$ boundary, in both the stationary and evolutive cases. In this setting, we also prove another result given by Imbert and Monneau in dimension one, namely that it is possible to use only a reduced class of test-functions.

1. Introduction

1.1. Hamilton–Jacobi equation and state constraint problems

Hamilton–Jacobi equations with state constraint boundary conditions naturally appear in optimal control problems in which trajectories must remain in a closure of a domain Ω. Introducing flux-limited solutions, Imbert and Monneau study a more general class of boundary conditions in [1] (respectively [2]) for one-dimensional (respectively multidimensional) spacial domains. They prove existence and uniqueness results for multi-junction domains for quasi-convex Hamiltonians. Recently Lions and Souganidis obtain a comparison principle for non-convex Hamiltonians in [3,4].

Let us recall Soner's formulation of state constraint (SC) problems, see [5,6]: a function $u \in C(\bar{\Omega})$ is a viscosity solution of the SC problem if it satisfies, in the viscosity sense

$$\begin{cases} u + H(\nabla u) = 0 & \text{in} \quad \Omega \\ u + H(\nabla u) \geq 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

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This formulation does not seem to take into account the boundary behavior of u as a sub-solution. However, in [5], Soner proves the uniqueness of bounded uniformly continuous solutions of (1) for a convex Hamiltonian H. Capuzzo-Dolcetta and Lions also prove the uniqueness of continuous solutions of (1) in [7] for a Hamiltonian which is non necessarily convex.

In [8], also for a convex Hamiltonian H, Ishii and Koike show the uniqueness of possibly discontinuous solutions for the problem (1) with the following extra inequation for subsolutions,

$$\begin{cases} u + H(\nabla u) = 0 & \text{in} \quad \Omega \\ u + H(\nabla u) \geq 0 & \text{on} \quad \partial\Omega \\ u + H_{in}(\nabla u) \leq 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

where H_{in} is an "inward Hamiltonian" [8].

In [1], for a quasi-convex and continuous coercive Hamiltonian H and Ω a bounded interval, Imbert and Monneau proved that the SC problem (1) is equivalent to the flux-limited problem,

$$\begin{cases} u + H(\nabla u) = 0 & \text{in} \quad \Omega \\ u + H^-(\nabla u) \geq 0 & \text{on} \quad \partial\Omega \\ u + H^-(\nabla u) \leq 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

where H^- is the nonincreasing part of the Hamiltonian along the inward vector which will be defined in Section 2. The function H^- is the same as H_{in} in [8] if H is convex and coercive. Imbert and Monneau also proved the result for the evolution type equation. In this paper, we prove the same result for a general $\mathcal{C}^1$ bounded open set Ω of $\mathbb{R}^d$.

In some cases, there are links between those three problems. We have the direct following implication: a solution of (3) is also a solution of (2) which is also a solution of (1). When the spatial domain is an open interval and with the same assumptions on H, Imbert and Monneau show that a function is a solution of (1) if and only if it is a solution of (3). We will prove the same result in the multidimensional setting. This result shows that the uniqueness obtained by Ishii and Koike for problem (2) implies the uniqueness of possibly discontinuous solutions of (1). However, this requires restrictive assumptions: a $\mathcal{C}^1$ domain, a coercive Hamiltonian and a weak continuity condition to be imposed to sub-solutions. The latter weak continuity assumption was already made in [9]. The main contribution of this article is two-fold: on the one hand, we can deal with equations of evolution type; on the other hand, the Hamiltonian is not necessarily convex.

There are other related works. For Hamilton–Jacobi equations posed on two domains separated by an interface where some transmission conditions are imposed, see [10,11]. In [12,13], the authors study an optimal control problem where the trajectories of the controlled system must remain in an embedded network.

In this paper, we also prove a useful result about flux-limited solutions in the multidimensional setting: the class of test functions can be significantly reduced. Indeed, this new class contains only test functions with one slope in space. The proof is essentially based on ideas from [1,2]. In particular it uses a property of critical slopes at the boundary for sub and supersolutions. To obtain this result for a multi-dimensional regular bounded open set, we introduce a local parametrization of the boundary. This result has many applications. For example, Proposition 2.16 in [1] allows to get weaker viscosity inequalities with test functions that do not need to be C^1. This is useful for proving convergence and error estimates for a numerical scheme in [14]. Also the important theorem of classification of effective boundary conditions, see [1, Theorem 1.1], is a consequence of the result concerning the use of reduced class of test-functions. The theorem of classification says that a general boundary condition is equivalent to a flux-limited one.

Organization of the paper. In Section 2, we give the definition of a flux-limited solution by introducing the non-increasing/non-decreasing part of a Hamiltonian along the normal direction on a boundary point.