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Some results for a class of quasilinear elliptic equations with singular nonlinearity

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ABSTRACT

In this paper, motivated by recent works on the study of the equations which model the electrostatic MEMS devices, we study the quasilinear elliptic equation involving a singular nonlinearity

$$\begin{cases} -(r^{\alpha}|u'(r)|^{\beta}u'(r))' = \dfrac{\lambda r^{\gamma} f(r)}{(1-u(r))^2}, & r \in (0,1), \\ 0 \le u(r) < 1, & r \in (0,1), \\ u'(0) = u(1) = 0. \end{cases}$$

According to the choice of the parameters α, β and γ, the differential operator which we are dealing with corresponds to the radial form of the Laplacian, the p-Laplacian and the k-Hessian. In this work we present conditions over which we can assert regularity for solutions, including the case $\lambda = \lambda^*$, where λ^* is a critical value for the existence of solutions. Moreover, we prove that whenever the critical solution is regular, there exists another solution of mountain pass type for λ close to the critical one. In addition, we use the Shooting Method to prove uniqueness of solutions for λ in a neighborhood of 0.

1. Introduction

In this work, we deal with the following class of singular elliptic problems

$$\begin{cases} -(r^{\alpha}|u'(r)|^{\beta}u'(r))' = \dfrac{\lambda r^{\gamma} f(r)}{(1-u(r))^2}, & r \in (0,1), \\ 0 \le u(r) < 1, & r \in (0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{P_λ}$$

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where α, β and γ are real constants and f is a continuous positive function over $(0, 1)$ satisfying

$$\beta > -1; \tag{H1}$$

$$0 < \int_0^1 r^\gamma f(r)\,\mathrm{d}r < +\infty \quad \text{for } r \in (0,1). \tag{H2}$$

Note that, according to the choice of the parameters α, β and γ, the operator $Lu := -(r^\alpha |u'|^\beta u')'$ corresponds to the radial form of different many operators, namely:

(i) $\alpha = n-1$	$\beta = 0$	$\gamma = n-1$	(Laplacian)
(ii) $\alpha = n-1$	$\beta = p-2$	$\gamma = n-1$	(p-Laplacian)
(iii) $\alpha = n-k$	$\beta = k-1$	$\gamma = n-1$	(k-Hessian) .

1.1. *Motivation and related results*

Singular problems of the form

$$-Lu = \lambda g(u), \tag{1.1}$$

for an elliptic operator L under various boundary conditions, have been extensively studied since the papers of D. Joseph and T. Lundgren [17], J. Keener and H. Keller [18] and M. Crandall and P. Rabinowitz [4,5].

It has been shown in these pioneering works that there exists a critical threshold $\lambda^* > 0$ such that (1.1) admits positive solutions for $0 < \lambda < \lambda^*$, while no positive solutions exist for $\lambda > \lambda^*$. In [19], F. Mignot and J-P. Puel studied regularity results to certain nonlinearities, namely, $g(u) = e^u, g(u) = u^m$ with $m > 1, g(u) = 1/(1-u)^k$ with $k > 0$. In this work, we are interested in a class of operators L which includes in particular the Laplacian, p-Laplacian and k-Hessian. We recall that the k-Hessian operator, usually represented by $S_k(D^2u)$, is defined as the sum of all principal $k \times k$ minors of the Hessian matrix D^2u. For instance $S_1(D^2u) = \Delta u$ and $S_N(D^2u) = \det D^2u$ is the Monge–Ampère operator.

In recent years considerable attention has been paid to a particular case of (1.1) as follows

$$\begin{cases} -\Delta u(x) = \dfrac{\lambda f(x)}{(1-u(x))^2}, & \text{in } \Omega, \\ 0 \le u(x) < 1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

motivated by its applicability to the modeling of electrostatic MEMS (*Micro Electro Mechanical Systems*). For works related with this class of problems and applications, we refer, among many others, to [6,8,9,11,12, 14,13,19] and references therein. Notice that when Ω is the unit ball in $\mathbb{R}^N$, and $\alpha = \gamma = n-1$ and $\beta = 0$, problem (1.2) corresponds to a particular case of (P_λ).

MEMS are micro-devices consisting of electrical and mechanical components combining together on a chip to produce a system of miniature dimensions (between 1 and 100 micrometers, that is 0,001 and 0,1 millimeters, respectively — less than the thickness of a human hair). They are essential components of the modern technology that is currently driving telecommunications, commercial systems, biomedical engineering and space exploration. For more information on the applications, development, etc. of the fundamental partial differential equations that model those devices, see D. Bernstein and J. Pelesko [22] and P. Esposito et al. [10]. The study of this class of problems has received much attention not just due to its applicability but also due to its mathematical importance as we can see in [11].

We also mention that D. Castorina et al. [2] studied, for $1 < p \le 2$, the p-MEMS equation (the MEMS equation for the p-Laplacian), that is,