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Viscosity effect on the degenerate lake equations

Bilal Al Taki*

Lebanese University, Laboratory of Mathematics- EDST and Faculty of Sciences I, Beirut, Lebanon
University of Savoy Mont Blanc, LAMA, UMR 5127 CNRS, 73376, Le Bourget du lac cedex, France

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ABSTRACT

This paper concerns the effect of viscosity on the degenerate lake equations (anelastic limit) when the bottom topography vanishes on the shore. We establish the existence and uniqueness of a global weak solutions for various choices of viscosity term in weighted Sobolev spaces where the weight is assumed to be a *power type weight*. Our solution is constructed by adapting carefully the known works on the incompressible Navier–Stokes equations in unweighted context. Finally we study, in one case, the vanishing viscosity limit when the solution of the inviscid lake equations is regular enough generalizing the results by Martino et al. (2001) to the degenerate case.

1. Introduction

The viscous lake equations (anelastic limit) have been derived in order to model the evolution of the vertically averaged horizontal components of the 3D velocity to the incompressible viscous fluid confined to a shallow basin with varying bottom topography. These equations can be obtained from the following viscous shallow water equations by letting the Froude number (noted by Fr in the sequel) go to zero when the initial height converges to a non-constant function depending on the space variable:

$$\begin{cases} \partial_t(hv) + \mathrm{div}(hv \otimes v) + A_h(v) + \dfrac{1}{2\,\mathrm{Fr}^2}\nabla h^2 = 0, & \text{in } \Omega, \\ \partial_t h + \mathrm{div}(hv) = 0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $h(t,x)$ is the water height, $v(t,x)$ is the velocity of a viscous fluid and A_h is a viscous second order operator depending on h. $v \otimes v$ is the matrix with components $v_i v_j$. h is classically assumed positive but can eventually vanishes, v is a vector valued function and both are defined on a bounded subset Ω of $\mathbb{R}^2$.

* Correspondence to: University of Savoy Mont Blanc, LAMA, UMR 5127 CNRS, 73376, Le Bourget du lac cedex, France.
E-mail address: bilal.al-taki@univ-smb.fr.

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Following Lions's book [29] (see also [1,8,9,32]), there exist various models (or approximations) for A_h like for instance

$$A_h(\cdot) = -2\mu \mathrm{div}(hD(\cdot) + h\mathrm{div}(\cdot)\,\mathbb{I}), \qquad A_h(\cdot) = -\mu h\Delta(\cdot), \tag{1.2}$$

where $\mu > 0$ represents the eddy viscosity coefficient, $\mathbb{I}$ is the 2×2 identity matrix and $D(\cdot) = (\nabla\cdot + (\nabla\cdot)^t)/2$ is the deformation tensor.

Before moving to viscous lake model, let us fix some ideas concerning the analysis of well posedness of System (1.1). One of the major difficulty on this analysis is to deal with vacuum. Since the works of P.-L. Lions on the compressible Navier–Stokes equations (the same equations where $\frac{1}{2\,\mathrm{Fr}^2}\nabla h^2$ is replaced by ∇h^γ, $\gamma > 0$ and $A_h = -\mu h\Delta(\cdot)$), we are able to prove the existence of global weak solutions of System (1.1) with $A_h = -\mu h\Delta(\cdot)$ for large initial data that may vanish. Nonetheless, important progress has been made by D. Bresch and B. Desjardins in [3] to handle the existence of global weak solutions of viscous shallow water equations with $A_h(\cdot) = -2\mu\mathrm{div}(hD(\cdot))$ as diffusion term. The key point in their paper [3] is to show that the structure of the diffusion term provides some regularity for h thanks to a new mathematical entropy inequality. In [3], the authors proved the existence of global weak solutions of System (1.1) whereas damping terms are added to the momentum equation $(1.1)_1$. This result was improved very recently by A. Vasseur, C. Yu in [42] where the authors proved that the result holds true without adding the damping terms. As a limitation of these elegant results [3,42], the global well posedness of viscous shallow water equations with $A_h(\cdot) = -2\mu\mathrm{div}(hD(\cdot) + h\mathrm{div}(\cdot)\,\mathbb{I})$ as diffusion term is still a very interesting open problem.

The viscous lake equations considered in this paper have the following equations

$$\begin{cases} \partial_t(bu^\mu) + \mathrm{div}(bu^\mu \otimes u^\mu) + A_b(u^\mu) + b\nabla p^\mu = 0, \quad \text{in } \Omega, \\ \mathrm{div}(bu^\mu) = 0, \quad \text{in } \Omega, \end{cases} \tag{1.3}$$

for $(x,t) \in \Omega \times (0,T)$ with $\Omega \subset \mathbb{R}^2$, a bounded Lipschitz domain. Here, $u^\mu = u^\mu(x,t) = (u_1^\mu(x,t), u_2^\mu(x,t))$ stands for the two-dimensional horizontal component of the fluid velocity, $p = p(t,x)$ is the pressure and A_b is a viscous second order operator depending on b which can satisfy the two expressions

$$\text{(i)}\ A_b(\cdot) = -2\mu\mathrm{div}(bD(\cdot) + 2b\mathrm{div}(\cdot)\,\mathbb{I}), \qquad \text{(ii)}\ A_b(\cdot) = -\mu b\Delta(\cdot).$$

Moreover, the bottom function $b(x)$ is a given function which assumed to be a *power type weight* in the sequel. The second equation in System (1.3) shows that the system does not describe incompressible flow, it is a constraint that plays a role similar to that played by the incompressibility condition for the incompressible Navier–Stokes system. However, since we do not have an existence result concerning the solution of viscous shallow water equations with viscosity term given by $A_h(\cdot) = -2\mu\mathrm{div}(hD(\cdot) + h\mathrm{div}(\cdot)\,\mathbb{I})$, then there is no mathematical result until now allowing us to justify the derivation of viscous lake model (1.3)-(i) from viscous shallow water model (1.1). The unique rigorous derivation (also in the non-degenerate case, i.e. when the bottom topography is strictly positive) is limited to the case when $A_h = -2\mu\mathrm{div}(2hD(\cdot))$ and can be found in the paper of D. Bresch, M. Gisclon and C. K. Lin [5]. Noticing that the difference between these expressions of viscosity term, namely $A_b = -\mu\mathrm{div}(2bD(\cdot))$ and $A_b = -2\mu\mathrm{div}(bD(\cdot) + b\mathrm{div}(\cdot)\,\mathbb{I})$, is not very important on the analysis of weak solutions of viscous lake model (1.3).

Constant bathymetry. In case when b is a constant, System (1.3) becomes similar to the classical 2D incompressible Navier–Stokes equations which is very well understood. In such case, the reader is referred to [28,39,40,2] for various existing surveys on the question.

Variable bathymetry. When the depth b varies but bounded away from zero, the well-posedness can be proved by adapting the same procedure as in C. D. levermore and M. Sammartino [27] without additional difficulty.