Nonlinear Analysis 148 (2017) 61–87

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

A Hölder estimate for non-uniform elliptic equations in a random medium

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ARTICLE INFO

Article history:
Received 3 March 2016
Accepted 16 September 2016
Communicated by S. Carl

MSC:
35J15
35J25
35J70
35R05
35R60

Keywords:
Random media
Conductivity
Stationary–ergodic
Realization
Diffeomorphism

ABSTRACT

Uniform regularity for second order elliptic equations in a highly heterogeneous random medium is concerned. The medium is separated by a random ensemble of simply closed interfaces into a connected sub-region with high conductivity and a disconnected subset with low conductivity. The elliptic equations, whose diffusion coefficients depend on the conductivity, have fast diffusion in the connected sub-region and slow diffusion in the disconnected subset. Without a stationary–ergodic assumption, a uniform Hölder estimate in $\omega, \epsilon, \lambda$ for the elliptic solutions is derived, where ω is a realization of the random ensemble, $\epsilon \in (0, 1]$ is the length scale of the interfaces, and $\lambda^2 \in (0, 1]$ is the conductivity ratio of the disconnected subset to the connected sub-region. Results show that if external sources are small enough in the disconnected subset, the uniform Hölder estimate in $\omega, \epsilon, \lambda$ holds in the whole domain. If not, it holds only in the connected sub-region. Meanwhile, the elliptic solutions change rapidly in the disconnected subset.

1. Introduction

In this article, we study the uniform regularity for second order elliptic equations in a highly heterogeneous random medium. The medium is separated by a random ensemble of simply closed surfaces of codimension one with jump type conductivity across the surfaces. This problem has many applications in physics and engineering, for example, contaminant transport in the subsurface, heat flow in random media, the stress in composite materials, and so on (see [11,19,20,29]).

Let $\epsilon \in (0, 1]$ represent the length scale of the surfaces separating the spatial domain, $Y \equiv [0, 1)^n$ be the unit cube in $\mathbb{R}^n$ for $n \geq 2$, Y_m be a smooth simply-connected sub-domain of Y with boundary ∂Y_m, $Y_f \equiv Y \setminus \overline{Y_m}$, $\mathbf{d}_1 \equiv \text{dist}(\partial Y, Y_m) > 0$, $(\Omega, \mathcal{F}, \mathcal{P})$ mean a probability space, $\Phi(\cdot, \omega) : \mathbb{R}^n \to \mathbb{R}^n$ denote a

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http://dx.doi.org/10.1016/j.na.2016.09.009

C^1-diffeomorphism for any $\omega \in \Omega$, $\mathcal{D}$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and

$$\begin{cases} \Upsilon(\cdot,\omega) \text{ denote the inverse mapping of } \Phi(\cdot,\omega), \\ \mathcal{D}(\epsilon) \equiv \{x \in \mathcal{D} |\ \mathrm{dist}(x, \partial\mathcal{D}) \geq \epsilon\}, \\ \mathcal{I}_{\omega,\epsilon} \equiv \{j \in \mathbb{Z}^n |\ \epsilon\Phi(Y + j, \omega) \subset \mathcal{D}(\epsilon)\}, \\ \Gamma_\omega^\epsilon \equiv \bigcup_{j \in \mathcal{I}_{\omega,\epsilon}} \epsilon\Phi(\partial Y_m + j, \omega), \\ \mathcal{D}_{\omega,m}^\epsilon \equiv \bigcup_{j \in \mathcal{I}_{\omega,\epsilon}} \epsilon\Phi(Y_m + j, \omega), \\ \mathcal{D}_{\omega,f}^\epsilon \equiv \mathcal{D} \setminus \overline{\mathcal{D}_{\omega,m}^\epsilon}. \end{cases} \tag{1.1}$$

From (1.1), we see that $\mathcal{D} = \mathcal{D}_{\omega,f}^\epsilon \cup \mathcal{D}_{\omega,m}^\epsilon \cup \Gamma_\omega^\epsilon$ is obtained from the image of a periodic medium under a random deformation followed by a rescaling, $\mathcal{D}_{\omega,f}^\epsilon$ is a connected sub-region of $\mathcal{D}$, and $\mathcal{D}_{\omega,m}^\epsilon$ is a disconnected subset of $\mathcal{D}$ for each $\omega \in \Omega$.

Let $\lambda^2 \in (0, 1]$ represent the conductivity ratio of the subset $\mathcal{D}_{\omega,m}^\epsilon$ to the sub-region $\mathcal{D}_{\omega,f}^\epsilon$. So the connected region $\mathcal{D}_{\omega,f}^\epsilon$ has high conductivity and the disconnected set $\mathcal{D}_{\omega,m}^\epsilon$ has low conductivity. Let $\mathbf{K}$ be a positive definite piecewise continuous matrix in $\mathbb{R}^n$. The problem that we consider is

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega,\lambda^2,\epsilon} \nabla U + Q) = F & \text{in } \mathcal{D}, \\ U = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{1.2}$$

where Q, F are known functions and $\mathbf{K}_{\omega,\lambda^2,\epsilon}$ is the conductivity with the expression

$$\mathbf{K}_{\omega,\lambda^2,\epsilon}(x) \equiv \begin{cases} \mathbf{K}\left(\Upsilon\left(\frac{x}{\epsilon}, \omega\right)\right) & \text{if } x \in \mathcal{D}_{\omega,f}^\epsilon \\ \lambda^2 \mathbf{K}\left(\Upsilon\left(\frac{x}{\epsilon}, \omega\right)\right) & \text{if } x \in \mathcal{D}_{\omega,m}^\epsilon \end{cases} \quad \text{for } \omega \in \Omega, \epsilon \in (0, 1], \lambda > 0.$$

Problem (1.2) is a non-uniform elliptic equation with random coefficients and random interfaces.

This setting of the random medium is introduced in [12,13], is a random perturbation of periodic structures, but is not a special case of the existing theories. Under a stationary–ergodic assumption, some new homogenization results can be obtained (see [12,13]). For more results on the medium of above setting, please see [4,5,10,15,12,13,27] and references therein. Our concern is the uniform regularity for second order elliptic equations on the medium of above setting. Different from the usual homogenization results on random media, our main result shows that, without a stationary–ergodic assumption, a Hölder estimate independent of the realization ω for the solution of (1.2) can be proved.

Whenever Q and F are smooth in $\mathcal{D}$ and small in $\mathcal{D}_{\omega,m}^\epsilon$, a piecewise smooth solution U of (1.2) exists uniquely and the H^1 norm of the solution in the connected sub-region $\mathcal{D}_{\omega,f}^\epsilon$ is bounded uniformly in $\epsilon, \lambda, \omega$ (see [21]). However, that is not the case for the solution U in the disconnected subset $\mathcal{D}_{\omega,m}^\epsilon$ and also the second order derivatives of U may not be bounded uniformly in $\epsilon, \lambda, \omega$ in the connected sub-region $\mathcal{D}_{\omega,f}^\epsilon$.

For periodic domain case (a special case of random media [19,24]), uniform regularity of the elliptic equations had been studied extensively. For example, uniform Hölder, $W^{1,p}$, and Lipschitz estimates in ϵ for uniform elliptic case of (1.2) (i.e., $\lambda = 1$) with Hölder coefficients were proved in [9,8]. Uniform $W^{1,p}$ estimate in ϵ for uniform elliptic case of (1.2) with continuous coefficients was shown in [14] and the same problem with VMO coefficients could be found in [26]. Uniform $W^{1,p}$ estimate for the Laplace equation in periodic perforated domains was considered in [22] and the same problem in Lipschitz estimate was studied in [25]. Uniform Hölder and Lipschitz estimates in λ, ϵ for non-uniform elliptic equations with discontinuous periodic coefficients were shown in [31].

In the modeling of random media, a stationary–ergodic assumption plays an important role because, after the homogenization process, the averaged model is deterministic (see [3,19,12,13]). In [7], a class of