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Homogeneous Einstein (α, β)-metrics on compact simple Lie groups and spheres

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A B S T R A C T

In this paper, we study homogeneous Einstein (α, β)-metrics on compact Lie groups and spheres. We first show that any left invariant Einstein (α, β)-metric on a connected compact simple Lie groups except SU(2) with vanishing S-curvature must be a Randers metric. Secondly, we prove that any Sp$(n+1)$-invariant Einstein (α, β)-metric on $S^{4n+3} (n \in \mathbb{N}^+)$ with vanishing S-curvature is either a Randers metric, or SU$(2n+2)$-invariant. Finally, we give a complete description of SU$(n+1)$-invariant Einstein Finsler metrics on $S^{2n+1} (n \geq 2)$.

1. Introduction

One of the most important problems in Riemann–Finsler geometry is the structure and classification of Einstein manifolds. Recall that a Finsler metric $F(x, y)$ on an n dimensional manifold M is called an Einstein metric if there exists a smooth function $\lambda(x)$ on M such that

$$\mathrm{Ric}(x, y) = \lambda(x) F^2(x, y).$$

The following problem, asked openly in many occasions by S.S. Chern, is still open:

Does every smooth manifold admit an Einstein Riemann–Finsler metric?

This problem is extremely involved and we believe it will be standing for a rather long period. However, the problem has motivated great interest of geometers and led to many results on Einstein Finsler metrics

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on manifolds. Up to now, most of the known Einstein Finsler metrics are either of Randers type or Ricci flat; see for example [3,4,7,13,21,23,27].

One effective approach to the above problem is to consider some special Finsler metrics. In this direction, invariant Einstein Finsler metrics on homogeneous manifolds are very interesting. Recall that in [29], W. Ziller classified homogeneous Einstein Riemannian metrics on spheres. Later, H. Wang, L. Huang and S. Deng [25] classified homogeneous Einstein–Randers metrics on spheres; see [8] for some further results on homogeneous Einstein–Randers metrics.

In this paper, we study homogeneous Einstein (α,β)-metrics on spheres. (α,β)-metrics form an important class of Finsler metrics. They are defined by a Riemannian metric $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and an 1-form $\beta=b_i(x)y^i$ in the form $F=\alpha\phi(\frac{\beta}{\alpha})$. It has been proved that $F=\alpha\phi(\frac{\beta}{\alpha})$ is a positive definite Finsler metric with $\|\beta\|_\alpha<b_0$ if and only if $\phi=\phi(s)$ is a positive C^∞ function on $(-b_0,b_0)$ satisfying the following condition:

$$\phi(s)-s\phi'(s)+(\rho^2-s^2)\phi''(s)>0,\quad |s|\le\rho<b_0. \tag{1.1}$$

The Randers metric $F=\alpha+\beta$ is just the special (α,β)-metric with $\phi(s)=1+s$. The function ϕ is called of Randers type if there exist constants k_1,k_2,k_3 such that $\phi(s)=k_1\sqrt{1+k_2s^2}+k_3s$. It is easy to see that F is a Randers metric if and only if ϕ is of Randers type.

In [28], L. Zhou first gave some formulas of the Riemann curvature and Ricci curvature for (α,β)-metrics. Later, X. Cheng, Z. Shen and Y. Tian [7] found some errors in his formulas. They then present the correct formulas of the Riemann curvature and Ricci curvature for (α,β)-metrics. They also show that if $\phi(s)$ is a polynomial in s, then the (α,β)-metric $F=\alpha\phi(\frac{\beta}{\alpha})$ is Einstein if and only if it is Ricci flat.

Recently, based on the formula of Ricci curvature for (α,β)-metrics in [7], we gave a formula of Ricci curvature for homogeneous (α,β)-metrics in [26]. Using this formula, we obtained a sufficient and necessary condition for a compact homogeneous (α,β)-metric to be Einstein and with vanishing S-curvature.

In the present paper, we shall continue our work in [26]. We first consider left invariant Einstein (α,β)-metrics on compact connected Lie groups, and prove the following main result.

Theorem A. *Let G be a compact connected, simply connected Lie group, and $F=\alpha\phi(\frac{\beta}{\alpha})$ be a left invariant Einstein (α,β)-metric on G with vanishing S-curvature. Then either ϕ is of Randers type, or $G=\mathrm{SU}(2)$.*

Then we study left invariant Einstein (α,β)-metrics on SU(2) with vanishing S-curvature, and give a complete description of such metrics; see Theorem 4.3 for the results.

Next, we study homogeneous Einstein (α,β)-metrics on spheres. There are two types of homogeneous (α,β)-metrics on spheres, one is $\mathrm{Sp}(n)$-invariant, and the other is $\mathrm{SU}(n)$-invariant. An unexpected result is the following

Theorem B. *Let $F=\alpha\phi(\frac{\beta}{\alpha})$ be an $\mathrm{Sp}(n+1)$-invariant Einstein metric on $S^{4n+3}(n\ge 1)$ with vanishing S-curvature. Then either ϕ is of Randers type, or F is $\mathrm{SU}(2n+2)$-invariant.*

Finally, we study $\mathrm{SU}(n+1)$-invariant Einstein (α,β)-metrics on S^{2n+1}, and obtain a sufficient and necessary condition for these metrics to be Einstein; see Theorem 5.1.

The arrangement of this paper is as the following. In Section 2, we present some preliminaries on Finsler geometry. In particular, we introduce the notion of Riemann curvature and S-curvature of a Finsler space. Section 3 is devoted to studying the conditions for a homogeneous (α,β)-metric to be Einstein. In Section 4, we study left invariant Einstein (α,β)-metrics on compact Lie groups, and give a proof of Theorem A. In Section 5, we study homogeneous (α,β)-metrics on spheres. In particular, we present a proof of Theorem B, and give a sufficient and necessary condition for an $\mathrm{SU}(n+1)$-invariant (α,β)-metric on S^{2n+1} to be Einstein.