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Boundary layer analysis of nonlinear reaction–diffusion equations in a polygonal domain

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ABSTRACT

We propose a boundary layer analysis which fits a domain with corners. In particular, we consider nonlinear reaction–diffusion problems posed in a polygonal domain having a small diffusive coefficient $\varepsilon > 0$. We present the full analysis of the singular behaviours at any orders with respect to the parameter ε where we use a systematic nonlinear treatment initiated in Jung et al. (2016). The boundary layers are formed near the polygonal boundaries and two adjacent ones overlap at a corner P and the overlapping produces additional layers, the so-called corner layers. It is noteworthy that the boundary layers are also degenerate due to the singularities of the solutions involving a negative power of the radial distance to the corner P which are present in the Laplace operator on a sector (sector corresponding to the part of the polygon near the corner). The corner layers are then designed to absorb both the singularities and the interaction of the two boundary layers at P.

1. Introduction

In real-life problems regarding partial differential equations(PDE), smooth domains are hardly found. The domains rather possess corners where two pieces of the boundary meet and their tangential derivatives on each piece are discontinuous. Understanding the behaviour of solutions near the corner or boundary singularities is important in e.g. aerodynamics, hydrodynamics and fracture mechanics (see e.g. [22,23]). Similarly, in the numerical analysis and implementation of numerical methods we often encounter domains with corners and hence the knowledge of the solution structures will be very useful in designing finite elements or finite volume schemes (see e.g. [3,5,8,17,19]). Although many methods and theories were developed for smooth domains, they cannot be directly applied to domains with singularities (see e.g. [22,12]). In addition, the regularity of the solutions to the PDE would be generally very limited.

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In this article we consider the following nonlinear singularly perturbed reaction–diffusion problems:

$$\begin{cases} -\varepsilon\Delta u^\varepsilon + g(u^\varepsilon) = f, & \text{in } \Omega, \\ u^\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a polygonal domain, either convex or non convex, $f = f(x, y)$ and $g = g(u)$ are given smooth functions with

$$g(0) = 0, \qquad g'(u) \geq \lambda > 0, \quad \forall u \in \mathbb{R}. \tag{1.2a}$$

For the existence and uniqueness of the solution to (1.1), we need to make assumptions on the growth at infinity of g: we assume that there exists a constant $c_1 > 0$ and an even number $p \geq 2$ such that

$$c_1(|u|^p - 1) \leq g(u)u \leq \frac{1}{c_1}(|u|^p + 1), \quad \forall u \in \mathbb{R}. \tag{1.2b}$$

We then find $u^\varepsilon \in H_0^1(\Omega) \cap L^p(\Omega)$ (see e.g. [21,24]). Throughout the paper, we assume that the functions f, g are smooth with the conditions (1.2).

For the existence we classically construct an approximate solution u_m using the Galerkin method. We obtain an a priori estimate on u_m in $H_0^1(\Omega) \cap L^p(\Omega)$ thanks to (1.2b) and then pass to the limit $m \to \infty$. The uniqueness is immediate because for $w = u_1 - u_2$ with u_1, u_2, solutions to (1.1), we have

$$-\varepsilon\Delta ww + \lambda w^2 \leq -\varepsilon\Delta(u_1 - u_2)w + (g(u_1) - g(u_2))w = 0. \tag{1.3}$$

Integrating over Ω we find that $w = 0$, that is $u_1 = u_2$ in $H_0^1(\Omega)$.

Remark 1.1. We note that u^ε belongs to $H^1(\Omega)$ which is enough for our analysis. For polygonal domains of the type considered, some additional regularity on u^ε can be obtained using [12]. Indeed, thanks to (1.2b), $g(u^\varepsilon) \in L^{p'}(\Omega)$ ($p' = p/(p-1)$; note that $p \geq 2$ and $1 < p' \leq 2$), so that by [12], u^ε belongs to $W^{2,p'}(\Omega)$. The regularity of u^ε is limited; for instance for $p = 2$ it is shown in [4,12] that $u^\varepsilon \notin H^3(\Omega)$ e.g. if the polygon Ω is convex. Some additional remarks on the regularity of u^ε on a general polygonal domain are given in Remark 2.3.

The reaction–diffusion equations are important in many applications like chemical and biological systems consisting of interacting components (see, e.g. [6,10,25,28,18,7,9,27,29]). In a fast reaction system, often observed in real applications, the diffusion coefficient ($\varepsilon > 0$ here) is relatively small and the system is singularly perturbed. Beside the intrinsic interest of considering a problem like (1.1) in a nonsmooth domain, this study can be seen as a step towards understanding more complex problems such as, in fluid mechanics, the so-called cavity driven flows [26,13,15].

In [18] we fully discussed the same problem (1.1) in a general smooth domain in which case the boundary layers vary exponentially in the direction of the inward normal to the boundary (See also [2,32]). In this article Ω denotes a polygonal domain; we assume that $\partial\Omega$ has N sides or edges, Γ_i, and N vertices, P_i, $i = 1, \ldots, N$. Let ω_i denote the interior angle of $\partial\Omega$ at P_i, measured counterclockwise from Γ_i to Γ_{i-1}, with $\Gamma_0 = \Gamma_N$ (see Fig. 3).

Then the solutions to problem (1.1) display both a corner singularity near the vertices P_i and the sharp transitions near the side Γ_i when ε is small and they produce two different types of boundary layers: ordinary boundary layers and elliptic corner layers. Problems of this type were studied in the pioneering articles [20,30], and more recently in [1,33,31]. See also [11,14,15,17] and [19]. In [20], the author obtained valid asymptotic expansions for the solutions of linear singularly perturbed reaction–diffusion equations on a polygonal domain and also obtained the estimates for the derivatives of the solutions. In [30], the authors obtained the same ones for linear convection–diffusion equations on a rectangular domain.

In the text below we will fully analyse the boundary and corner layers. To handle the nonlinear term $g(u^\varepsilon)$, we will expand g as we did in [18].