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Critical exponent for nonlinear wave equations with frictional and viscoelastic damping terms

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ABSTRACT

In this paper, we study the Cauchy problem for a nonlinear wave equation with frictional and viscoelastic damping terms. Our aim is to obtain the threshold, to classify the global existence of solutions for small initial data or the finite time blow-up of solutions, under the growth condition of the nonlinearity.

1. Introduction

In this paper we consider the Cauchy problem for a wave equation with two types of damping terms

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u - \Delta \partial_t u = f(u), & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \qquad \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u_0(x)$ and $u_1(x)$ are given functions, and concerning the nonlinearity $f(u)$ we shall deal with only the typical case such as $f(r) := |r|^p$ $(p > 1)$ without loss of the essence.

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After A. Matsumura [29] has established pioneering basic decay estimates to the linear equation

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = 0, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

many mathematicians have concentrated on solving a typical important nonlinear problem of the semi-linear damped wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = |u|^p, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

and in that case we necessarily remind of the Fujita critical exponent. That is, there exists a real number $p_F \in (1,\infty)$ such that if $p > p_F$, then for some range of p the corresponding Cauchy problem (1.3) has a small global in time solution $u(t,x)$ for small initial data $[u_0,u_1]$, while in the case when $p \in (1,p_F]$, the corresponding problem under the condition $\int_{\mathbb{R}^n} u_i(x)dx > 0$ for $i = 0,1$ does not have any nontrivial global solutions. The number p_F is called as the Fujita critical exponent, and nowadays it is well-known that $p_F = 1 + \frac{2}{n}$. Even if we restricted to the Cauchy problem case in $\mathbb{R}^n$, one can cite so many related research papers due to [10–12,14,17,21,23–25,28,30,32,39,41,44] and the references therein. It should be emphasized that the first success to find out the Fujita exponent in a complete style for all $n \geq 1$ is in the work due to Todorova–Yordanov [41]. Anyway, these results are based on an important recognition that the asymptotic profile as $t \to +\infty$ of the solution of the linear equation (1.2) is a constant multiple of the Gauss kernel or a solution of the corresponding heat equation with an appropriate initial data. This type of diffusion phenomenon is discussed in [3,4,6,9,12,13,19,23,24,28,30–32,35,36,38,40,42] and the references therein.

On the other hand, if the problem (1.3) is replaced by the following strongly damped wave equation case,

$$\begin{cases} \partial_t^2 u - \Delta u - \Delta\partial_t u = \mu f(u), & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

there seem not so many related research papers at present. In the case when the linear equations are concerned with $\mu = 0$ in (1.4), one has two pioneering papers due to Ponce [33] and Shibata [37], in which they studied $L^p - L^q$ decay estimates of the solutions to (1.4) with $\mu = 0$. Quite recently, Ikehata–Todorova–Yordanov [22] and Ikehata [15,16] have caught an asymptotic profile of solutions to problem (1.4) with $\mu = 0$, and in fact, its profile is the so-called diffusion wave, which is well-studied in the field of the Navier–Stokes equation case. In this case, an oscillation property occurs in the low frequency region, while in the usual frictional damping case (1.2) one cannot observe any such oscillation properties in the low frequency part. There is a big difference between (1.2) and (1.4) with $\mu = 0$. In connection with this, several decay estimates for wave equations with structural damping, which interpolate (1.3) and (1.4) are extensively studied to the equation

$$\partial_t^2 u - \Delta u + b(t)(-\Delta)^\theta \partial_t u = \mu f(u), \tag{1.5}$$

where $\theta \in [0,1]$ and $\mu \geq 0$ in the papers due to [2,4–6,18,27,26], [34] ($\mu = 0$, $b(t) \equiv 1, \theta = 0$, exterior domain case), [43] ($\mu = 0, \theta = 0$), and the references therein. At this stage, if one considers the original problem (1.1), which has two types of damping terms, a natural question arises that which is dominant as $t \to +\infty$, frictional damping or viscoelastic one? About this fundamental question, quite recently Ikehata–Sawada [20] have studied the problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u - \Delta\partial_t u = 0, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$