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A new phenomenon in the critical exponent for structurally damped semi-linear evolution equations

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ABSTRACT

In this paper, we find the critical exponent for global small data solutions to the Cauchy problem in $\mathbb{R}^n$, for dissipative evolution equations with power nonlinearities $|u|^p$ or $|u_t|^p$,

$$u_{tt} + (-\Delta)^\delta u_t + (-\Delta)^\sigma u = \begin{cases} |u|^p, \\ |u_t|^p. \end{cases}$$

Here $\sigma, \delta \in \mathbb{N} \setminus \{0\}$, with $2\delta \leq \sigma$. We show that the critical exponent for each of the two nonlinearities is related to each of the two possible asymptotic profiles of the linear part of the equation, which are described by the diffusion equations:

$$v_t + (-\Delta)^{\sigma-\delta} v = 0,$$
$$w_t + (-\Delta)^\delta w = 0.$$

The nonexistence of global solutions in the critical and subcritical cases is proved by using the test function method (under suitable sign assumptions on the initial data), and lifespan estimates are obtained. By assuming small initial data in Sobolev spaces, we prove the existence of global solutions in the supercritical case, up to some maximum space dimension $\bar{n}$, and we derive L^q estimates for the solution, for $q \in (1, \infty)$. For $\sigma = 2\delta$, the result holds in any space dimension $n \geq 1$. The existence result also remains valid if σ and/or δ are fractional.

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1. Introduction

In this paper, we look for the critical exponents for global small data solutions to

$$\begin{cases} u_{tt} + (-\Delta)^\delta u_t + (-\Delta)^\sigma u = |u|^p, & t \geq 0, x \in \mathbb{R}^n, \\ (u, u_t)(0, x) = (u_0, u_1)(x), \end{cases} \tag{1}$$

with $\sigma \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{N}$, and to

$$\begin{cases} u_{tt} + (-\Delta)^\delta u_t + (-\Delta)^\sigma u = |u_t|^p, & t \geq 0, x \in \mathbb{R}^n, \\ (u, u_t)(0, x) = (u_0, u_1)(x), \end{cases} \tag{2}$$

with $\sigma, \delta \in \mathbb{N} \setminus \{0\}$. When $2\delta \leq \sigma$, we prove that these critical exponents are, respectively,

$$p_0 := 1 + \frac{2\sigma}{(n - 2\delta)_+}, \tag{3}$$

$$p_1 := 1 + \frac{2\delta}{n}. \tag{4}$$

By critical exponent we mean that suitable global small data solutions exist in the supercritical case, whereas global solutions cannot exist, under suitable sign assumption on the data, in the critical and subcritical cases. The term $(-\Delta)^\delta u_t$ represents a damping term. When $\delta > 0$, the damping is sometimes said to be *structural* (or strong). In the case $n \leq 2\delta$, the notation $p_0 = \infty$ in (3) (see Notation 4 in Section 1.3) denotes that the nonexistence result holds for (1) for any $p > 1$.

Exponents (3) and (4) are easily found by homogeneity arguments when $2\delta \leq \sigma$, whereas the same arguments lead to the exponents $1 + 2\sigma/(n - \sigma)$ and $1 + \sigma/n$, respectively, for (1) and (2), when $2\delta \geq \sigma$. Indeed, by using a quite standard test function method, we prove that global, weak, solutions cannot exist, under suitable sign assumption on the data, for critical and subcritical powers, in all these cases (Theorems 1 and 2), and we prove some lifespan estimates for the local solutions.

On the other hand, it is well-known that existence of global small data solutions may not be proved in the whole supercritical range, for some partial differential equations, as a counterpart of a nonexistence result related to homogeneity arguments. For instance, the critical exponent $1 + \sqrt{2}$ for the existence of global small data solutions to the semilinear wave equation $u_{tt} - \Delta u = |u|^p$ in space dimension $n = 3$ (see [26]) is strictly greater than the critical exponent 2 found by homogeneity arguments [28] (see [18,19], and the reference therein, for the existence exponent in higher space dimension).

By the converse, in 2001, G. Todorova and B. Yordanov [45] proved global existence of small data solution for the semilinear damped wave equation ($\sigma = 1$ and $\delta = 0$ in (1)),

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & t \geq 0, x \in \mathbb{R}^n, \\ (u, u_t)(0, x) = (u_0, u_1)(x), \end{cases} \tag{5}$$

in the supercritical range $p > 1 + 2/n$, by assuming small data in weighted energy space. Here $1 + 2/n$ is Fujita exponent, obtained by homogeneity arguments (see, in particular, [46]). By only assuming data in Sobolev spaces, the existence result was proved in space dimension $n = 1, 2$ in [24], by using energy methods, and in space dimension $n \leq 5$ in [37], by using $L^r - L^q$ estimates, $1 \leq r \leq q \leq \infty$.

Indeed, the main difference with respect to the wave equation with no dissipation, is that the damping term u_t in (5) produces the *diffusion phenomenon*. This effect modifies the asymptotic profile of the solution to the corresponding linear problem so that it can be described by the solution to a heat equation with suitable initial data (see [22] and, later, [21,29,39]).

We mention that, recently, the first author, together with S. Lucente and M. Reissig [11,13], studied a wave equation with time-dependent dissipation, for which the existence exponent coincides with Fujita exponent in space dimension $n = 1, 2$, and it is larger than this latter in (odd) space dimension $n \geq 3$.