Nonlinear Analysis 149 (2017) 81–110

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Neumann problem for the nonlinear Klein–Gordon equation

Ivan Naumkin

Université Pierre et Marie Curie, Laboratoire Jacques-Louis Lions, B.C. 187, 4 place Jussieu, 75252 Paris Cedex 05, France

ARTICLE INFO

Article history:
Received 1 December 2015
Accepted 13 October 2016
Communicated by Enzo Mitidieri

MSC:
35M13
35B40
35A01
35A02

Keywords:
Nonlinear Klein–Gordon equation
Neumann problem
Initial–boundary value problem
Scattering-critical

ABSTRACT

The Neumann initial–boundary value problem for the nonlinear Klein–Gordon equation

$$\begin{cases} v_{tt} + v - v_{xx} = \mu v^3, \ \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ v(0,x) = v_0(x), \ v_t(0,x) = v_1(x), \ x \in \mathbb{R}^+, \\ (\partial_x v)(t,0) = h(t), \ \ t \in \mathbb{R}^+, \end{cases} \tag{0.1}$$

for real μ, $v_0(x), v_1(x)$ and $h(t)$, is considered. We prove the global well-posedness for the initial–boundary value problem (0.1) and we present a sharp time decay estimate of the solution in the uniform norm. Also we study the asymptotic behavior of the solution to (0.1). We show that the cubic nonlinearity in the Neumann initial–boundary value problem (0.1) is scattering-critical.

1. Introduction

We study the Klein–Gordon equation with cubic nonlinearity and inhomogeneous Neumann initial–boundary value conditions

$$\begin{cases} v_{tt} + v - v_{xx} = \mu v^3, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ v(0,x) = v_0(x), v_t(0,x) = v_1(x), \quad x \in \mathbb{R}^+, \\ (\partial_x v)(t,0) = h(t), \quad t \in \mathbb{R}^+, \end{cases} \tag{1.1}$$

where $\mathbb{R}^+ = (0,\infty)$ is the positive half line, the constant $\mu \in \mathbb{R}$ and $v_0(x),\ v_1(x)$ and $h(t)$ are real-valued functions.

The nonlinear Klein–Gordon equation appears in the study of several problems of mathematical physics. For example, this equation arises in general relativity, nonlinear optics (e.g., the instability phenomena such

E-mail address: ivan.naumkin@upmc.fr.

http://dx.doi.org/10.1016/j.na.2016.10.014

as self-focusing), plasma physics, fluid mechanics, radiation theory or spin waves [5,7,12]. When an external force is applied to a system, frequently this force is introduced into the equation in the form of inhomogeneous boundary condition. Thus, the initial–boundary value problems are often called forced problems. In terms of applications, the initial–boundary value problems seem to be more natural, than the initial value problems (Cauchy problems). However, they are much less studied than the Cauchy problems. In part, this is because of the mathematical difficulties that appear in the study of the initial–boundary problems. In the present paper we prove the global in time existence of small solutions to the initial–boundary value problem (1.1) with inhomogeneous Neumann boundary conditions and we present a sharp time decay estimate for the solutions in the $\mathbf{L}^{\infty}$ norm. Also we obtain the large time asymptotics for the solutions to the initial–boundary value problem (1.1).

There are various papers that considered the global well-posedness of the Cauchy problem for the nonlinear Klein–Gordon equation in different space dimensions and for different types of nonlinearities. We mention, for example, the works [3,4,9–11,13–16,18,22,25], and the references cited there. As far as we know there are no results on the well-posedness of the initial–boundary value problem (1.1). We fill this void in the present paper.

The scattering theory is based on the comparison of the perturbed dynamics with the free one. In the case of the nonlinear scattering problem, one compares the large time behavior of the solutions to nonlinear equations with the asymptotics of the solutions to the corresponding linear equations. For a general study of the nonlinear scattering problem we refer to [19–21,23,24]. In the nonlinear scattering theory, the nonlinearity can be considered as a perturbation of the linear equation and one can establish an analogy of the nonlinear scattering theory with the linear potential scattering theory (see, for example, [17]). Then, there appears the notion of scattering-critical nonlinearity or long range scattering. That is, a nonlinearity with the same large time behavior as the linear part of the equation. The effect of such nonlinearity on the equation is no longer negligible and the solutions are affected by this nonlinearity, i.e., there appear some corrections in the large time behavior of the solutions. In particular, this means that there are no asymptotically free solutions (that have the same asymptotics for large time as the solutions to the corresponding linear equation). Cubic nonlinearities in one space dimension are often scattering-critical. For example, there appears a logarithmic correction in the asymptotics of the solution to the Schrödinger equation in one space dimension with gauge invariant nonlinearity of power 3 [17]. This is also the case of the one dimensional Cauchy problem for the nonlinear Klein–Gordon equation with the nonlinearity μv^3 [3,13,9]. To our knowledge, there are no results neither on the sharp time decay nor on the asymptotics of solutions to the initial–boundary value problem (1.1). In the present work we fill this gap.

At first glance one might want to reduce (1.1) to a Cauchy problem by an even extension approach. However, in this way we cannot answer the well-posedness questions. Besides, the known results for the Cauchy problem require more smoothness for the extended initial data ($\mathbf{H}^{4,1}$ in paper [9] versus $\mathbf{H}^{2,1}$ by our method). Moreover, the extension method is not general (most of equations and boundary conditions cannot be solved by the extension approach).

One of the particularities of the initial–boundary value problems is the fact that the well-posedness of the problem depends on the boundary data. One needs to answer the question about the quantity of the boundary conditions that are required to guarantee the existence of a unique solution. Besides, one also has to deal with the compatibility conditions between the initial and boundary data in order to study the well-posedness of the given initial–boundary value problem. The quantity and the form of the compatibility conditions are related to the type of the boundary problem. For example, it may happen that for some Neumann initial–boundary value problem no compatibility conditions are required, and, at the same time, some compatibility conditions are needed for the well-posedness of the Dirichlet problem. In the case of the problem (1.1), we prove that it is enough to ask for one boundary and one compatibility condition to ensure the existence of a unique solution.