Nonlinear Analysis 149 (2017) 117–129

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

$L^{p(\cdot),\lambda(\cdot)}$ regularity for fully nonlinear elliptic equations

Lin Tang

LMAM, School of Mathematical Sciences, Peking University, Beijing, 100871, PR China

ARTICLE INFO

Article history:
Received 8 September 2016
Accepted 15 October 2016
Communicated by Enzo Mitidieri

MSC:
35J60
35B65
42B25

Keywords:
Fully nonlinear elliptic equation
Viscosity solution
Muckenhoupt weight
Morrey space

ABSTRACT

We establish the variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}$ estimate to the Dirichlet problem for fully nonlinear elliptic equations on a $C^{1,1}$ bounded domain for variable exponents $p(\cdot)$ and $\lambda(\cdot)$.

1. Introduction

This paper is concerned with the following problem for fully nonlinear elliptic equation

$$\begin{cases} F(D^2u, Du, u, x) = f & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. Here, $F = F(X, z, s, t)$ is a real valued Carathéodory function defined on $S(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega$, where $S(n)$ is the set of $n \times n$ real symmetric matrices ordered in the usual way: $X \geq 0$ when $\langle X\xi, \xi\rangle \geq 0$ for all $\xi \in \mathbb{R}^n$, where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product, and $Y \geq X$ means $Y - X \geq 0$. We assume that F is uniformly elliptic with ellipticity constants λ and Λ, that is, there exist constants λ and Λ with $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda\|Y\| \leq F(X + Y, z, s, x) - F(X, z, s, x) \leq \Lambda\|Y\|, \tag{1.2}$$

for all $X, Y \in S(n)$, $Y \geq 0$, $z \in \mathbb{R}^n, s \in \mathbb{R}$ and almost all $x \in \Omega$, where $\|Y\| := \sup_{|x|=1} |Yx|$ that is equal to the maximum eigenvalue of Y where $Y \geq 0$.

E-mail address: tanglin@math.pku.edu.cn.

http://dx.doi.org/10.1016/j.na.2016.10.016
0362-546X/

L. Caffarelli [7] developed a new approach based on the Pucci–Aleksandrov inequality, leading to $W^{2,p}$ regularity for solution to (1.1) in the range $p > n$. By using weak reverse Hölder inequalities, L. Escauriaza [16] extended the results in [7] to the range $p > n - \epsilon$ with a small $\epsilon > 0$ depending on the ellipticity constants of the nonlinear operator considered. Employing the techniques from [7,16], N. Winter [23] derived boundary $W^{2,p}$-a priori estimates for the solutions of (1.1), and proved $W^{2,p}$-solvability results as well. Recently, S. Byun et al., in [5] extended the results of Winter [23] to the settings of weighted Sobolev spaces.

On the other hand, in recent years there has been an increasing interest in the study of various mathematical problems with variable exponents. For examples of these physical applications, we refer to homogenization theory of strongly anisotropic materials [24,26], electrorheological fluids [20,21], temperature dependent viscosity fluids [25] and image restoration [9]. In general, physical situations with strong anisotropy are well described by the variable exponent spaces. This leads us to the study of partial differential equations in the setting of variable exponent Morrey spaces. There have been rich research activities on regularity estimates for elliptic and parabolic problems in the frame of variable exponent function spaces, see [1,2,4–6] and references therein.

Our goal is to prove that, under appropriate hypotheses on the data, for each variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ there exists a unique strong solution $u \in L^{p(\cdot),\lambda(\cdot)}(\Omega)$ of (1.1) that satisfies the estimate

$$\|u\|_{W^{2,p(\cdot),\lambda(\cdot)}(\Omega)} \le c\|f\|_{L^{p(\cdot),\lambda(\cdot)}(\Omega)}$$

with a positive constant c independent of f.

In fact, even in the special case $\lambda(\cdot) \equiv 0$, our results are also new. Thus, we prove that $f \in L^{p(\cdot),\lambda(\cdot)}(\Omega)$ implies $D^2u \in L^{p(\cdot),\lambda(\cdot)}(\Omega)$, by the properties of functions with variable exponent Morrey spaces regular gradients, which leads to better integrability and even variable exponent Hölder continuity of the gradient of u (see Corollary 3.1).

The organization of this paper is as follows. In Section 2, we introduce the variable exponent Morrey space, list the hypotheses on the nonlinearity F and the weight ω, and state our result. In Section 3, we prove the regularity in variable exponent Morrey space of the second derivatives of solutions to (1.1), and the corresponding finer smoothing of the gradient.

2. Assumptions and main result

We start this section with some standard notations and definitions. For a point $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and real number $r > 0$, let $B_r(y) = \{x \in \mathbb{R}^n : |x - y| < r\}$. For a function $u : \mathbb{R}^n \to \mathbb{R}$, we denote the gradient of u by $Du = (D_1u, \dots, D_nu)$, and its Hessian by $D^2u = (D_{ij}u)$, where $D_iu = D_{x_i}u = \frac{\partial u}{\partial x_i}$, $D_{ij}u = D_{x_ix_j}u = \frac{\partial^2 u}{\partial x_i \partial x_j}$ for $i, j = 1, \dots, n$.

Now, let us discuss the structure conditions to be imposed on $F : S(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$. Let $0 < \lambda \le \Lambda$. We introduce the Pucci extremal operators $\mathcal{P}^+$, $\mathcal{P}^-$ associated with λ, Λ that are defined as follows: for $X \in S(n)$,

$$\mathcal{P}^-(X, \lambda, \Lambda) := \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i \quad \text{and} \quad \mathcal{P}^+(X, \lambda, \Lambda) := \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i,$$

where e_i are the eigenvalues of X.

We introduce the structure condition that will be frequently used in this paper as follows: F is nonincreasing in s, $F(0, 0, 0, s) = 0$, and

$$\begin{aligned} \mathcal{P}^-(X - Y, \lambda, \Lambda) - k_1|z - \widetilde{z}| - k_2|s - \widetilde{s}| &\le F(x, z, s, x) - F(Y, \widetilde{z}, \widetilde{s}, z) \\ &\le \mathcal{P}^+(X - Y, \lambda, \Lambda) + k_1|z - \widetilde{z}| + k_2|s - \widetilde{s}| \end{aligned} \tag{2.1}$$