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On a singular periodic Ambrosetti–Prodi problem

Alessandro Fonda[a], Andrea Sfecci[b,*]

[a] Università degli Studi di Trieste, Dipartimento di Matematica e Geoscienze, P.le Europa, 1, I-34127 Trieste, Italy

[b] Università Politecnica delle Marche, Dipartimento di Ingegneria Industriale e Scienze Matematiche, Via Brecce Bianche, 12, I-60131 Ancona, Italy

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ABSTRACT

We investigate the possibility of extending a classical multiplicity result by Fabry, Mawhin and Nkashama to a periodic problem of Ambrosetti–Prodi type having a nonlinearity with possibly one or two singularities. In the second part of the paper we study the existence of periodic rotating solutions for radially symmetric systems with nonlinearities of the same type.

1. Introduction

In 1972, Ambrosetti and Prodi [1] obtained a multiplicity result for the solutions of a Dirichlet problem associated to an elliptic equation, which can be said to have influenced the research in the field of boundary value problems up to the present days.

Let us recall the result of [1], as refined by Berger and Podolak in [3], by writing the Dirichlet problem as

$$\begin{cases} \Delta u + h(u) = s\varphi_1(x) + w(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$, while $\varphi_1(x)$ is the positive eigenfunction associated to the first eigenvalue λ_1 of the Laplacian, with Dirichlet boundary conditions, and $w(x)$ is a suitably smooth function. Assuming $h : \mathbb{R} \to \mathbb{R}$ to be twice continuously differentiable and strictly convex, with

$$0 < h'(-\infty) < \lambda_1 < h'(+\infty) < \lambda_2,$$

* Corresponding author.
E-mail addresses: a.fonda@units.it (A. Fonda), sfecci@dipmat.univpm.it (A. Sfecci).

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(where λ_2 is the second eigenvalue), they proved the existence of an $s_0 \in \mathbb{R}$ such that

- if $s < s_0$, there are no solutions,
- if $s = s_0$, there is exactly one solution,
- if $s > s_0$, there are exactly two solutions.

Since then, many variants and generalizations have been proposed, see e.g. [2,4,6,15–17,19–21,23–25,28], a far from being exhaustive list. Remarkably, the name *Ambrosetti–Prodi problem* remained attached to all such situations when a multiplicity result structure as the one described above appears.

Searching for an analogue for the periodic problem, Fabry, Mawhin and Nkashama [7] considered in 1986 the second order differential equation

$$x'' + f(x)x' + h(t,x) = s. \qquad (E_s)$$

(In this case, the Laplacian is replaced by a second derivative, and the first eigenvalue associated to the periodic problem is equal to zero.) They were able to prove the following Ambrosetti–Prodi type of result.

Theorem 1.1 (*Fabry–Mawhin–Nkashama*)**.** *Assume $f : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ to be continuous functions, with T-periodicity in the t variable. If*

$$\lim_{|x|\to\infty} h(t,x) = +\infty, \quad \text{uniformly in } t \in [0,T],$$

then there exists an $s_0 \in \mathbb{R}$ such that

- *if $s < s_0$, there are no T-periodic solutions,*
- *if $s = s_0$, there is at least one T-periodic solution,*
- *if $s > s_0$, there are at least two T-periodic solutions.*

We will take the above theorem as our starting point, and develop some possible generalizations. In the first part of the paper we focus our attention on the case when the nonlinearities in Eq. (E_s) are defined only for x varying in an open interval (a,b) of $\mathbb{R}$, with possibly one or two singularities. Here is our result, extending Theorem 1.1 to such a situation.

Theorem 1.2. *Assume $f : (a,b) \to \mathbb{R}$ and $h : \mathbb{R} \times (a,b) \to \mathbb{R}$ to be continuous functions, with T-periodicity in the t variable, such that*

$$\lim_{x\to a^+} h(t,x) = \lim_{x\to b^-} h(t,x) = +\infty, \quad \text{uniformly in } t \in [0,T]. \qquad (1)$$

If $b = +\infty$, the same conclusion of Theorem 1.1 for Eq. (E_s) holds. On the other hand, if $b < +\infty$, the same is true assuming, in addition, that

$$f(x) \geq -\eta \quad \text{and} \quad h(t,x) \geq h_m(x), \quad \text{for every } x \in (a,b),$$

where η is a positive constant and $h_m : (a,b) \to \mathbb{R}$ is continuous and such that

$$\int_c^b h_m(x)\,dx = +\infty, \qquad (2)$$

for some $c \in (a,b)$.

A few comments on the above statement are in order. Notice that, in the case $(a,b) = \mathbb{R}$, Theorem 1.2 reduces to Theorem 1.1. If $b = +\infty$, no assumptions besides the continuity are required on the function f. When $b < +\infty$, the repulsive singularity at $x = b$ has to be sufficiently strong so to ensure that the solutions of (E_s) cannot collide with it. On the contrary, it is remarkable that the attractive singularity at $x = a$ does not require an assumption of this type.