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On the convergence of solutions to bilateral problems with the zero lower constraint and an arbitrary upper constraint in variable domains

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ABSTRACT

In this article, we give sufficient conditions for the convergence of minimizers and minimum values of integral and more general functionals on sets of functions defined by bilateral constraints in a sequence of domains Ω_s contained in a bounded domain Ω of $\mathbb{R}^n$ $(n \geqslant 2)$. We study the case where the lower constraint is zero and the upper constraint is an arbitrary nonnegative measurable function on Ω. The statements of our main results include the condition of the Γ-convergence of the functionals (defined on the spaces $W^{1,p}(\Omega_s)$) to a functional defined on $W^{1,p}(\Omega)$ and the condition of the strong connectedness of the spaces $W^{1,p}(\Omega_s)$ with the space $W^{1,p}(\Omega)$, where $p > 1$. At the same time, because of the specificity of the imposed constraints, the exhaustion condition of the domain Ω by the domains Ω_s and the proposed requirement on the behavior of the integrands of the principal components of the considered functionals are also important for our convergence results.

1. Introduction

In this article, for a given bounded domain Ω of $\mathbb{R}^n$ $(n \geqslant 2)$ and a given sequence of domains Ω_s contained in Ω, we consider a sequence of functionals $J_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ of the structure $J_s = F_s + G_s$, where $p > 1$, $\{F_s\}$ is a sequence of integral functionals whose integrands satisfy certain convexity and growth conditions, and $\{G_s\}$ is a sequence of weakly continuous functionals. Along with this, we consider the sequence of the sets $V_s(\psi) = \{v \in W^{1,p}(\Omega_s) : 0 \leqslant v \leqslant \psi \text{ a.e. in } \Omega_s\}$, where $\psi : \Omega \to \overline{\mathbb{R}}$ is a measurable function such that $\psi \geqslant 0$ a.e. in Ω. We study the question of the convergence of minimizers and minimum values of the functionals J_s on the sets $V_s(\psi)$. More exactly, we are interested in conditions for the convergence of these

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minimizers and minimum values to the corresponding minimizer and minimum value of a similar bilateral problem.

An analogous question was studied by the author in [13,21] for variational problems with the sets of constraints $V_s(\varphi,\psi)=\{v\in W^{1,p}(\Omega_s):\varphi\leqslant v\leqslant\psi \text{ a.e. in } \Omega_s\}$, where $\varphi,\psi\in W^{1,p}(\Omega)$. In these works, the conditions for the convergence of solutions to the given bilateral problems include the requirement of the Γ-convergence of the corresponding functionals to a functional defined on $W^{1,p}(\Omega)$ and the requirement of a certain (strong) connectedness of the spaces $W^{1,p}(\Omega_s)$ with the space $W^{1,p}(\Omega)$. At the same time, in [13], we assumed that $\psi-\varphi\geqslant\alpha$ a.e. in Ω for some constant $\alpha>0$ and, in [21], we assumed that the difference $\psi-\varphi$ satisfies the weaker condition $\psi-\varphi>0$ a.e. in Ω.

In the present article, in particular, the Γ-convergence of the sequence $\{F_s\}$ to a functional $F:W^{1,p}(\Omega)\to\mathbb{R}$, a certain convergence of the sequence $\{G_s\}$ to a functional $G:W^{1,p}(\Omega)\to\mathbb{R}$, and the strong connectedness of the spaces $W^{1,p}(\Omega_s)$ with the space $W^{1,p}(\Omega)$ are required to prove the convergence of minimizers and minimum values of the functionals J_s on the sets $V_s(\psi)$ to a minimizer and the minimum value of the functional $F+G$ on the set $V(\psi)=\{v\in W^{1,p}(\Omega):0\leqslant v\leqslant\psi \text{ a.e. in } \Omega\}$. At the same time, we assume that, for every $s\in\mathbb{N}$, the integrand $f_s:\Omega_s\times\mathbb{R}^n\to\mathbb{R}$ of the functional F_s satisfies the inequality

$$c_1|\xi|^p-\mu_s(x)\leqslant f_s(x,\xi)\leqslant c_2|\xi|^p+\mu_s(x)$$

for almost every $x\in\Omega_s$ and for every $\xi\in\mathbb{R}^n$, where c_1 and c_2 are preassigned positive constants and $\mu_s\in L^1(\Omega_s)$, $\mu_s\geqslant 0$ in Ω_s.

It should be noted that, because of the specificity of the bilateral problems under consideration (the lower constraint is zero and the upper constraint is an arbitrary nonnegative function on Ω), the following two conditions are also important for our convergence results. The first one is the next exhaustion condition of the domain Ω by the domains Ω_s:

$$\text{for every increasing sequence } \{m_j\}\subset\mathbb{N},\qquad \operatorname{meas}\Big(\Omega\setminus\bigcup_{j=1}^{\infty}\Omega_{m_j}\Big)=0. \tag{1.1}$$

The second condition is as follows:

$$\|\mu_s\|_{L^1(\Omega_s)}\to 0. \tag{1.2}$$

Concerning the notion of strong connectedness of a sequence of Sobolev spaces and the notion of Γ-convergence of functionals with variable domains of definition used in the present article, additionally to [13,21], we refer the reader, for instance, to [11,12,18,19]. We remark that the notion of strong connectedness of Sobolev spaces goes back to [10], where the condition of strong connectedness of n-dimensional domains was introduced. This condition can be considered as a prototype of the mentioned notion of strong connectedness of Sobolev spaces. Concerning the notion of Γ-convergence of functionals with the same domain of definition and the corresponding results, see, for instance, [6,8,30,31].

We note that, with the use of the techniques of the Γ-convergence theory, the asymptotic behavior (as $s\to\infty$) of solutions to variational problems for a quadratic integral functional with bilateral constraints of the type $\varphi_s\leqslant v\leqslant\psi_s$, where $\varphi_s,\psi_s:\mathbb{R}^n\to\overline{\mathbb{R}}$, was studied in [4]. At the same time, in view of the general character of these constraints, the corresponding limit variational problem also has a quite general form.

We also mention some works on the convergence of solutions of variational inequalities with bilateral constraints. In [25], it was shown that the G-convergence of a sequence of linear continuous operators $\mathcal{A}_s:\mathring{W}^{1,2}(\Omega)\to W^{-1,2}(\Omega)$ in divergence form to an operator $\mathcal{A}:\mathring{W}^{1,2}(\Omega)\to W^{-1,2}(\Omega)$ of the same kind implies the weak convergence of solutions of variational inequalities with the operators $\mathcal{A}_s$ and with the set of constraints $K(\psi_1,\psi_2)=\{v\in\mathring{W}^{1,2}(\Omega):\psi_1\leqslant v\leqslant\psi_2 \text{ a.e. in } \Omega\}$ to a solution of the corresponding variational inequality with the operator $\mathcal{A}$ and with the same set of constraints. At the same time, it was