Nonlinear Analysis 147 (2016) 96–109

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Global existence blow up and extinction for a class of thin-film equation☆

Qingwei Li, Wenjie Gao, Yuzhu Han*

School of Mathematics, Jilin University, Changchun 130012, PR China

ARTICLE INFO

Article history:
Received 7 June 2016
Accepted 26 August 2016
Communicated by S. Carl

MSC:
primary 35J15
secondary 35J20
35J70

Keywords:
Thin-film equation
Potential wells
Global existence
Uniqueness
Blow up

ABSTRACT

In this paper we use the modified method of potential wells to study the properties of solutions for a class of higher order nonlinear parabolic equations with p-Laplace term $-(|u_x|^{p-2}u_x)_x$ and nonlocal source $|u|^{q-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-1}u\mathrm{d}x$. Global existence, uniqueness, blow up in finite time and asymptotic behavior of solutions will be proved under different initial conditions. Furthermore, a numerical example is given to illustrate the blow-up of solutions in finite time.

1. Introduction

In this article, we consider the following initial and boundary value problem for a class of nonlinear parabolic equation describing thin-film epitaxial growth

$$\begin{cases} u_t + u_{xxxx} - (|u_x|^{p-2}u_x)_x = |u|^{q-1}u - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega |u|^{q-1}u\mathrm{d}x, & (x,t) \in \Omega \times (0,T), \\ u_x = u_{xxx} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}$ is an open interval, $p > 1, q > \max\{1,\ p-1\}$. $u_0 \in H^2(\Omega)$ satisfies $\int_\Omega u_0\mathrm{d}x = 0$.

It is well known that fourth-order reaction–diffusion equations describe a variety of important physical processes, such as phase transition, thin-film theory, lubrication theory etc. Usually Problem (1.1) can

☆ The project is supported by NSFC (11271154, 11401252), by Science and Technology Development Project of Jilin Province (20150201058NY, 20160520103JH) and by the project of The Education Department of Jilin Province (2015-463).

* Corresponding author.
E-mail addresses: qingweili_jlu@163.com (Q. Li), wjgao@jlu.edu.cn (W. Gao), yzhan@jlu.edu.cn (Y. Han).

http://dx.doi.org/10.1016/j.na.2016.08.021
0362-546X/

be regarded as a simplified model to describe the evolution of the epitaxial growth of nanoscale thin films (see [2,16]), where $u(x,t)$ denotes the height from the surface of the film, u_{xxxx} corresponds to the capillarity-driven surface diffusion, and the p-Laplace term $(|u_x|^{p-2}u_x)_x$ denotes the upward hopping of atoms. Due to Zangwill [16] the height $u(x,t)$ of a film in epitaxial growth can be described by the basic model

$$u_t = g - \nabla \cdot j + \eta \tag{1.2}$$

with periodic boundary conditions and some initial conditions. The phenomenological approach is to expand j in ∇u and its powers. Keeping only "sensible" terms yields

$$j = A_1\nabla u + A_2\nabla(\Delta u) + A_3|\nabla u|^2\nabla u + A_4\nabla|\nabla u|^2,$$

see [16] for details. Ortiz et al. [9] modified this model in several respects. In particular, they showed that A4 = 0 if Onsager's reciprocity relations held, and the terms in (1.2) had the following physical interpretations

$$\begin{aligned} g:&\quad \text{the deposition flux,}\\ A_1\Delta u:&\quad \text{diffusion due to evaporation–condensation,}\\ A_2\Delta^2 u:&\quad \text{capillarity-driven surface diffusion,}\\ A_3|\nabla u|^2\nabla u:&\quad \text{upward hopping of atoms.}\end{aligned}$$

The epitaxial growth of nanoscale thin films has recently received increasing interest in materials science. The following well-known equation

$$u_t + \Delta^2 u - \operatorname{div}(f(\nabla u)) = g(x), \tag{1.3}$$

where a reasonable choice of $f(s)$ is $f(s) = |s|^{p-2}s$ was studied by King, Stein and Winkler [2] who proved the existence, uniqueness and regularity of solutions in an appropriate function space for the initial and boundary value problem. Liu [6,7] studied the following equation

$$u_t + \operatorname{div}[m(u)k\nabla\Delta u - |\nabla u|^{p-2}\nabla u] = 0 \tag{1.4}$$

in one and two dimensional spaces. On the basis of the uniform Schauder type estimates and Campanato spaces, he proved the global existence of classical solutions. In the absence of $-(|u_x|^{p-2}u_x)_x$ in (1.1) Qu and Zhou [11] considered the following thin-film equation

$$u_t + u_{xxxx} = |u|^{p-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{p-1}u\,\mathrm{d}x. \tag{1.5}$$

By using the method of potential wells they obtained a threshold result of global existence and blow up for the sign-changing weak solutions and the conditions under which the global solutions extinct in finite time.

The method of potential well was first proposed by Sattinger [12] in 1968 when dealing with a class of non-linear hyperbolic initial–boundary value problem

$$\begin{cases} u_{tt} - \nabla^2 u + f(x,u) = 0, & (x,t) \in \Omega\times(0,T),\\ u = 0, & (x,t)\in \partial\Omega\times(0,T),\\ u(x,0) = U(x), \qquad u_t(x,0) = V(x) & x\in \Omega. \end{cases} \tag{1.6}$$

This method is based on the 'potential' energies $J(u)$ associated with Problem (1.6). Suppose that J has a local minimum at $u = U(x)$. Then, in analogy with the local minimum of a potential function for a mechanical system with a finite number of degree of freedom, imagine a potential well W situated at $u = U$ in function space. If U lies in W and if the total energy of the initial data is less than the depth of W, then Problem (1.6) has a global solution. After that many authors [1,4,5,8,10,13] studied the global existence and nonexistence of solutions of initial and boundary value problem for various nonlinear evolution equations