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Very weak solutions of the stationary Navier–Stokes equations for an incompressible fluid past obstacles

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ABSTRACT

We consider the stationary motion of an incompressible Navier–Stokes fluid past obstacles in $\mathbb{R}^3$, subject to the given boundary velocity v_b, external force $f = \operatorname{div} F$ and nonzero constant vector ke_1 at infinity. Our main result is the existence of at least one very weak solution v in $ke_1 + L^3(\Omega)$ for arbitrary large $F \in L^{3/2}(\Omega) + L^{12/7}(\Omega)$ provided that the flux of $v_b - ke_1$ on the boundary of each body is sufficiently small with respect to the viscosity ν. The uniqueness of very weak solutions is proved by assuming that F and $v_b - ke_1$ are suitably small. Moreover, we establish weak and strong regularity results for very weak solutions. In particular, our existence and regularity results enable us to prove the existence of a weak solution v satisfying $\nabla v \in L^{3/2}(\Omega)$.

1. Introduction

Let Ω be a smooth domain in $\mathbb{R}^3$, exterior to the union of a finite number of disjoint compact bodies $\mathcal{B}_1, \ldots, \mathcal{B}_\gamma$. The steady motion of a viscous incompressible fluid in Ω is governed by the Dirichlet problem for the Navier–Stokes equations:

$$\begin{cases} -\nu\Delta v + \operatorname{div}(v \otimes v) + \nabla\pi = f & \text{in } \Omega \\ \operatorname{div} v = 0 & \text{in } \Omega \\ v = v_b & \text{on } \partial\Omega \\ v(x) \to ke_1 & \text{as } |x| \to \infty. \end{cases} \tag{NS}$$

Here $v = (v^1, v^2, v^3)$ and π denote the unknown velocity and pressure fields of the fluid, respectively, while $f = \operatorname{div} F$, $F = (F_i^j)_{i,j=1,2,3}$ and $v_b = (v_b^1, v_b^2, v_b^3)$ denote the given external force and boundary velocity, respectively. Moreover, $ke_1 = (k, 0, 0) \neq 0$ is the prescribed constant vector at infinity and $\nu > 0$ is the given viscosity constant.

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The first aim of this paper is to obtain existence and uniqueness results for very weak solutions of the exterior Navier–Stokes problem (NS). The notion of very weak solutions of the stationary Navier–Stokes equations on a bounded smooth domain D has been introduced by Marušić-Paloka [21], Galdi–Simader–Sohr [10], Farwig–Galdi–Sohr [6], Kim [12] and Amrouche–Rodríguez Bellido [2] in order to prove an existence result for the Navier–Stokes problem with very irregular boundary data. Among others, Marušić-Paloka [21] firstly proved the existence of very weak solutions in $L^3(D)$ of the Navier–Stokes equations for arbitrary large data $F \in L^2(D)$ and $v_b \in L^2(\partial D)$ satisfying the compatibility condition $\int_{\partial D} v_b \cdot N\, dS = 0$, and then further generalized and improved by Kim [12] for the data $F \in L^{3/2}(D)$ and $v_b \in W^{-1/3,3}(\partial D)$.

Very weak solutions of the exterior Navier–Stokes problem (NS) can be defined similarly as in [21,10,6,12]. Then extending the results in [21,10,6,12] to exterior domains, we shall establish the existence of very weak solutions in $ke_1 + L^3(\Omega)$ of (NS) provided that the flux of $v_b - ke_1$ through the boundary of each body $\mathcal{B}_i$ is sufficiently small in comparison with the viscosity constant ν, i.e.,

$$\frac{1}{\nu}\sum_{i=1}^{\gamma}\left|\left\langle v_b - ke_1, N\right\rangle_{\partial\mathcal{B}_i}\right| \quad \text{is sufficiently small.} \tag{1}$$

The uniqueness of very weak solutions is also proved for suitably small data F and $v_b - ke_1$; see Theorem 2.1 in Section 2 for details.

The first fundamental contribution to the existence problem for (NS) was made by Leray [19] in 1933, who showed that if $F \in L^2(\Omega)$ and $v_b = 0$ on $\partial\Omega$, there exists at least one solution v having the finite Dirichlet integral, i.e., $\nabla v \in L^2(\Omega)$. Since his pioneer work, the authors in [7,20,3,8,11] have investigated the existence, uniqueness, regularity and asymptotic behavior of solutions for (NS). In particular, it was shown in [8] that if $F \in L^2(\Omega)$ and $v_b \in W^{1/2,2}(\partial\Omega)$, then there exists at least one solution v satisfying $v - ke_1 \in L^6(\Omega)$ and $\nabla v \in L^2(\Omega)$ when the data $v_b - ke_1$ satisfies the smallness condition (1). To the best of our knowledge, the existence problem with arbitrary flux at the boundary is one of the most outstanding open questions in the mathematical fluid mechanics even if Ω is a general bounded smooth domain in $\mathbb{R}^3$. Hence our smallness condition (1) seems to be an appropriate assumption for obtaining the existence of very weak solutions.

The second aim of this paper is to establish regularity results for very weak solutions v of (NS). Analogous regularity results can be found in [12] for bounded domains. The regularity result [12, Theorem 3] on a bounded domain D gives the existence of at least one weak solution in $W^{1,r}(D)$ of (NS), where $F \in L^r(D)$, $v_b \in W^{1-1/r,r}(\partial D)$, $3/2 \le r < \infty$, and $\int_{\partial D} v_b \cdot N = 0$. This existence result improves Serre's existence result [23]. In the same vein, our regularity result also enables us to prove the existence of a weak solution $v \in L^3(\Omega)$ satisfying $\nabla v \in L^{3/2}(\Omega)$ of (NS) provided that $F \in L^{3/2}(\Omega)$ and $v_b \in W^{1/3,3/2}(\partial\Omega)$; see Corollary 2.3 in Section 2. It is well-known that the norms $\|\nabla v\|_{3/2}$ and $\|v\|_3$ are invariant under the scaling $v_\lambda(x) = \lambda v(\lambda x)$ for each $\lambda > 0$. Hence, from the viewpoint of scaling invariance, our solution space plays a particular role in investigating the asymptotic behavior of the Navier–Stokes equations. On the other hand, the regularity result [11, Theorem 2.1] on an exterior domain shows that every weak solution v of (NS) for $F \in L^2(\Omega)$ necessarily satisfies $v - ke_1 \in L^4(\Omega)$. This gives an affirmative answer to Leray's open question, whether every weak solution satisfies the generalized energy equality. Another consequence of this regularity result is the existence of a weak solution v satisfying $\nabla v \in L^{3/2}(\Omega)$ of (NS) for the data $F \in L^2(\Omega) \cap L^{3/2}(\Omega)$. Hence our result, Corollary 2.3, reveals that the restriction $F \in L^2(\Omega)$ is indeed unnecessary from the viewpoint of the existence issues.

The corresponding existence problem to the case $k = 0$ is much more difficult because the linearized problem of (NS), i.e., the Stokes problem is not uniquely solvable in $L^3(\Omega)$. To get around this difficulty, Kozono and Yamazaki [18] introduce the Lorentz space $L^{q,r}(\Omega)$. Using the linearized method in such a Lorentz space, they constructed a solution $v \in L^{3,\infty}(\Omega)$ with $\nabla v \in L^{3/2,\infty}(\Omega)$ of (NS) for small data F in $L^{3/2,\infty}(\Omega)$. However, it remains still open to prove the uniqueness of weak solutions v satisfying $v \in L^{3,\infty}(\Omega)$