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Sobolev–Morrey regularity of solutions to general quasilinear elliptic equations

Sun-Sig Byun[a], Dian K. Palagachev[b,*], Pilsoo Shin[c]

[a] *Seoul National University, Department of Mathematical Sciences and Research Institute of Mathematics, Seoul 151-747, Republic of Korea*

[b] *Politecnico di Bari, Dipartimento di Meccanica, Matematica e Management, Via E. Orabona 4, 70125 Bari, Italy*

[c] *Seoul National University, Department of Mathematical Sciences, Seoul 151-747, Republic of Korea*

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ABSTRACT

We deal with the Dirichlet problem for general quasilinear elliptic equations over Reifenberg flat domains. The principal part of the operator supports natural gradient growth and its x-discontinuity is of small-BMO type, while the lower order terms satisfy controlled growth conditions with x-behaviour modelled by Morrey spaces. We obtain a Calderón–Zygmund type result for the gradient of the weak solution by proving that the solution gains Sobolev–Morrey regularity from the data of the problem.

1. Introduction

The regularity problem is one of the central topics in the general theory of PDEs. Its main goal is to establish how the smoothness of the data of a given differential problem influences the regularity of a solution, obtained under very general circumstances. Once having better smoothness, powerful tools of functional analysis apply to infer finer properties of the solution and the problem itself. The importance

* Corresponding author.
E-mail addresses: byun@snu.ac.kr (S.-S. Byun), dian.palagachev@poliba.it (D.K. Palagachev), shinpilsoo@snu.ac.kr (P. Shin).

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of these issues is even more evident if dealing with discontinuous differential operators over domains with non-smooth boundaries when many of the classical analysis techniques fail.

In the present paper we study the regularity problem in Sobolev–Morrey spaces for quasilinear divergence form elliptic equations. We are interested in obtaining an optimal Calderón–Zygmund type theory in such spaces under minimal assumptions to impose on the discontinuous nonlinearities and on the non-smooth underlying domain. We deal, precisely, with the Dirichlet problem

$$\begin{cases} \operatorname{div}\left(\mathbf{a}(x,u,Du)+\mathbf{b}(x,u)\right)=c(x,u,Du) & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

over bounded n-dimensional domains Ω and where the nonlinear terms $\mathbf{a}$, $\mathbf{b}$ and c are given by suitable Carathéodory maps.

We suppose that the generally non-smooth boundary of Ω is sufficiently flat in the sense of Reifenberg that means, roughly speaking, $\partial\Omega$ is well approximated by hyperplanes at each point and at each scale. This is a sort of "minimal boundary regularity" ensuring the validity of the main geometric analysis results in Ω, and it has proved to be a natural assumption to be required on $\partial\Omega$ when dealing with regularity problems for divergence form PDEs. In particular, C^1-smooth and Lipschitz continuous boundaries (with small Lipschitz constant) belong to that class, but the category of Reifenberg flat domains extends beyond these common examples and contains sets with rough fractal boundaries such as the Helge von Koch snowflake (see [24]).

The principal part $\mathbf{a}(x,u,Du)$ of the differential operator is supposed to be elliptic, measurable in x and it supports natural gradient growth, that is, $\mathbf{a}(x,u,Du)$ behaves as $|Du|^{m-1}$ with $m>1$. The most notable, by now classical, example is given by the m-Laplacian $|Du|^{m-2}Du$, but our results apply also to $\mathbf{A}(x,u)|Du|^{m-2}Du$ with a suitable elliptic matrix $\mathbf{A}$ and to more general operators which do not possess necessarily a variational structure. As for the lower order terms $\mathbf{b}$ and c, these are subject to controlled growth conditions

$$|\mathbf{b}(x,u)|\le \mathcal{O}\left(\varphi(x)+|u|^{\frac{m^*(m-1)}{m}}\right),$$
$$|c(x,u,Du)|\le \mathcal{O}\left(\psi(x)+|u|^{m^*-1}+|Du|^{\frac{m(m^*-1)}{m^*}}\right),$$

with the Sobolev conjugate m^* of m, and suitable Lebesgue integrable functions φ and ψ. It is worth noting that the growth requirements on the nonlinearities in (1.1) are indispensable in order to give sense of the concept of a weak solution $u\in W_0^{1,m}(\Omega)$ to (1.1), but these are very far from being sufficient to ensure better integrability of the gradient than that in $L^m(\Omega)$.

The importance of studying discontinuous problems of the type (1.1) over rough domains is justified by the fact that these arise naturally in mathematical models of real-world systems over media with fractal geometry such as blood vessels, composite materials and semiconductor devices, the internal structure of lungs, clouds, bacteria growth, optimal control of stock markets and the economic applications of the non-smooth variational analysis (see [17,20]). The x-discontinuity of $\mathbf{a}(x,u,Du)$, instead, could be regarded to crack ruptures of the media, such as small multipliers of the Heaviside step function for instance.

We are interested here of the case when φ and ψ control the x-behaviour of $\mathbf{b}$ and c in terms of the Morrey spaces $L^{p,\lambda}(\Omega)$ and $L^{q,\mu}(\Omega)$, respectively, with exponents satisfying $(m-1)p+\lambda>n$ and $mq+\mu>n$. Assuming that the discontinuity with respect to x in $\mathbf{a}(x,u,Du)$ is measured in terms of smallness of the bounded mean oscillation (BMO) seminorm, and that $\partial\Omega$ is Reifenberg flat, we prove that the problem (1.1) supports the Calderón–Zygmund property in the Sobolev–Morrey functional scales. In other words, the gradient of each $W_0^{1,m}(\Omega)$ weak solution to (1.1) gains better Lebesgue and Morrey integrability from φ and ψ, both strictly defined by n,m and the exponents p,λ,q and μ. As a consequence, Hölder continuity up to the boundary follows for the weak solution with optimal exponent expressed in terms of n,m,p,λ,q and μ.