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Viscosity solutions of second order integral–partial differential equations without monotonicity condition: A new result

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ABSTRACT

We show existence and uniqueness of a continuous with polynomial growth viscosity solution of a system of second order integral–partial differential equations (IPDEs for short) without assuming the usual monotonicity condition of the generator with respect to the jump component as in Barles et al.'s article (Barles et al., 1997). The Lévy measure is arbitrary and not necessarily finite. In our study the main tool we used is the notion of backward stochastic differential equations with jumps.

1. Introduction

The main objective of this paper is to deal with the following system of integral–partial differential equations: $\forall i \in \{1, \ldots, m\}$,

$$\begin{cases} -\partial_t u^i(t,x) - b(t,x)^\top D_x u^i(t,x) - \frac{1}{2}\mathrm{Tr}\big(\sigma\sigma^\top(t,x)D^2_{xx}u^i(t,x)\big) - Ku^i(t,x) \\ \quad - h^{(i)}(t,x,(u^j(t,x))_{j=1,m},(\sigma^\top D_x u^i)(t,x), B_i u^i(t,x)) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^k; \\ u^i(T,x) = g^i(x) \end{cases} \tag{1.1}$$

where the operators B_i and K are defined as follows:

$$\begin{aligned} B_i u^i(t,x) &= \int_E \gamma^i(t,x,e)\left(u^i(t,x+\beta(t,x,e)) - u^i(t,x)\right)\lambda(de) \quad \text{and} \\ Ku^i(t,x) &= \int_E \left(u^i(t,x+\beta(t,x,e)) - u^i(t,x) - \beta(t,x,e)^\top D_x u^i(t,x)\right)\lambda(de) \end{aligned} \tag{1.2}$$

where λ is a Lévy measure on $E := \mathbb{R}^\ell - \{0\}$ which integrates the function $(1 \wedge |e|^2)_{e\in E}$.

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The second order system of equation (1.1) is of non-local type since the operators $B_i u^i$ and Ku^i at (t,x) involve the values of u_i in the whole space $\mathbb{R}^k$ and not only locally, i.e. in a neighbourhood of (t,x).

This system of IPDEs, introduced by Barles et al. in [2], is deeply related to the following multidimensional backward stochastic differential equation (BSDE for short) with jumps whose solution, for fixed $(t,x)\in[0,T]\times\mathbb{R}^k$, is a triple of adapted stochastic processes $(Y_s^{t,x},Z_s^{t,x},U_s^{t,x})_{s\le T}$ with values in $\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2(\lambda)$ which mainly satisfy: $\forall i\in\{1,\dots,m\}$,

$$\begin{cases}-dY_s^{i;t,x}=h^{(i)}\left(s,X_s^{t,x},(Y_s^{j;t,x})_{j=1,m},Z_s^{i;t,x},\int_E\gamma_i(s,X^{t,x}{}_s,e)U_s^{i;t,x}(e)\lambda(de)\right)ds\\ \qquad -Z_s^{i;t,x}dB_s-\int_E U_s^{i;t,x}(e)\tilde{\mu}(ds,de),\quad \forall\, s\le T;\\ Y_T^{i;t,x}=g^i(X_T^{t,x}),\end{cases} \tag{1.3}$$

where:

(i) $B:=(B_s)_{s\le T}$ is a d-dimensional Brownian motion, μ an independent Poisson random measure with compensator $ds\lambda(de)$ and $\tilde{\mu}(ds,de):=\mu(ds,de)-ds\lambda(de)$ its compensated random measure;

(ii) for any $(t,x)\in[0,T]\times\mathbb{R}^k$, $(X_s^{t,x})_{s\le T}$ is the solution of the following standard stochastic differential equation of diffusion–jump type, i.e.,

$$X_s^{t,x}=x+\int_t^s b(r,X_r^{t,x})dr+\int_t^s\sigma(r,X_r^{t,x})dB_r+\int_t^s\int_E\beta(r,X_{r-}^{t,x},e)\tilde{\mu}(dr,de),$$
$$\text{for } s\in[t,T] \text{ and } X_s^{t,x}=x \text{ if } s\le t. \tag{1.4}$$

Actually it has been shown in [2] that, under standard assumptions on the functions b, σ, β, g^i, $h^{(i)}$ and γ_i and due to the Markovian framework of randomness which stems from the Markov process $X^{t,x}$, there exist deterministic continuous functions $(u^i(t,x))_{i=1,m}$ such that for any $s\in[t,T]$,

$$Y_s^{i;t,x}=u^i(s,X_s^{t,x}),\quad \forall i=1,\dots,m. \tag{1.5}$$

Moreover if for any $i=1,\dots,m$,

(a) $\gamma_i\ge 0$;

(b) the mapping $q\in\mathbb{R}\mapsto h^{(i)}(t,x,y,z,q)$ is non-decreasing, when the other components (t,x,y,z) are fixed;

then the functions $(u^i)_{i=1,m}$ is the unique continuous viscosity solution of system (1.1) in the class of functions with polynomial growth (at least). Conditions (a)–(b), which will be referred as the monotonicity conditions, are needed in [2] in order to have the comparison property and to treat the operator $B_i u^i$ which is not well-defined for an arbitrary u. However we should point out those conditions are not required in order to show the existence and uniqueness of the solution $(Y^{t,x},Z^{t,x},U^{t,x})$ of BSDE (1.3).

Therefore the main issue is to deal with the viscosity solutions of system (1.1) without assuming the above conditions (a)–(b) neither on γ_i nor on $h^{(i)}$, $i=1,\dots,m$. A step forward in the resolution of this problem is made by Hamadène–Morlais in [8] where it is shown that, when the Lévy measure λ is finite i.e. $\lambda(E)<\infty$, then system (1.1) has a unique solution which is given by the functions $(u^i)_{i=1,m}$ defined in (1.5).

The main objective of this paper is once more to deal with the problem of existence and uniqueness of a viscosity solution of system of IPDEs (1.1) without assuming the monotonicity conditions neither on γ_i nor on $h^{(i)}$, $i=1,\dots,m$ and for an arbitrary Lévy measure λ without assuming its finiteness as in [8]. There are two crucial points. The first one is the characterization (1.6) of the process $U^{t,x}=(U^{i;t,x})_{i=1,m}$ of the solution of the BSDE (1.3) by means of the functions $(u^i)_{i=1,m}$ defined in (1.5) and the jump–diffusion