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Multiple radial positive solutions of semilinear elliptic problems with Neumann boundary conditions

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ABSTRACT

Let B_R be a ball of radius R in $\mathbb{R}^N$. We analyze the positive solutions to the problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } B_R, \\ \partial_\nu u = 0, & \text{on } \partial B_R, \end{cases}$$

that branch out from the constant solution $u = 1$ as p grows from 2 to $+\infty$. The nonzero constant positive solution is the unique positive solution for p close to 2. We show that there exist arbitrarily many positive solutions as $p \to \infty$ (in particular, for supercritical exponents) or as $R \to \infty$ for any fixed value of $p > 2$, partially answering a conjecture in Bonheure et al. (2012). We give explicit lower bounds for p and R so that a given number of solutions exist. The geometrical properties of those solutions are studied and illustrated numerically. Our simulations motivate additional conjectures. The structure of the least energy solutions (among all or only among radial solutions) and other related problems are also discussed.

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1. Introduction

In this paper, we consider the semilinear elliptic problem

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{P}_{\lambda,p}$}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, $\lambda > 0$, $p > 2$ and ∂_ν denotes the outward normal derivative. This problem, sometimes referred to as the Lane–Emden equation with Neumann boundary conditions, arises for instance in mathematical models which aim to study pattern formation, and more specifically in those governed by diffusion and cross-diffusion systems [50]. The problem is also related to the stationary Keller–Segel system in chemotaxis [30,34,35,39].

As $(\mathscr{P}_{\lambda,p})$ admits a constant solution, the solvability of $(\mathscr{P}_{\lambda,p})$ differs from the case of positive solutions of the Lane–Emden equation with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = |u|^{p-2}u, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

for which it is well known that, if Ω is starshaped and $N \geqslant 3$, existence is restricted to the subcritical range

$$p < 2^* := \frac{2N}{N-2} \tag{1.2}$$

as a consequence of Pohozaev's identity (see [57]). In the sequel of the paper we set $2^* := +\infty$ if $N = 2$.

The subcriticality assumption (1.2) allows to tackle the problem $(\mathscr{P}_{\lambda,p})$ with variational methods, i.e., the equation arises as the Euler–Lagrange equation of the energy functional

$$\mathscr{E}_{\lambda,p} : H^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 + \lambda u^2 - \frac{1}{p}\int_\Omega |u|^p.$$

Moreover, due to the compact embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$, the existence of a solution to $(\mathscr{P}_{\lambda,p})$ follows by standard arguments. Indeed, it is enough to minimize $\mathscr{E}_{\lambda,p}$ on the Nehari manifold

$$\mathscr{N}_{\lambda,p} := \left\{ u \in H^1 \setminus \{0\} : \mathscr{E}'_{\lambda,p}(u)[u] = 0 \right\}$$

and to observe that the minimizer is nonnegative whereas the strong maximum principle implies its positivity. The minimizers are called least energy or ground state solutions. Looking at the quadratic form $\mathscr{E}''_{\lambda,p}(u_0)[u,u]$, it is easily seen that any minimizer u_0 is nonconstant if[1] $\lambda(p-2) > \lambda_2(\Omega)$. On the other hand, if λ is small, the only minimizer is the constant solution as Lin, Ni and Takagi [39] proved that uniqueness holds for $(\mathscr{P}_{\lambda,p})$ for λ small.

In contrast to the nonexistence result for (1.1), the energy functional for the critical exponent, $\mathscr{E}_{\lambda,2^*}$, achieves its minimum on $\mathscr{N}_{\lambda,2^*}$. Moreover, Wang [65] proved that when λ is sufficiently large, the constant solution cannot be a minimizer.

For λ small and $p = 2^*$, Lin and Ni [38] conjectured that the constant solution must be the unique solution. The conjecture was studied by Adimurthi and Yadava [1,2] and Budd, Knapp and Peletier [18] in the case of radial solutions when Ω is a ball. It happens that in this case, the conjecture is true in dimension $N = 3$ or $N \geqslant 7$, while it is false in dimension $N = 4, 5, 6$. The conjecture was further extended to convex domains in dimension $N = 3$ and has lead to many developments in the recent years. We refer to [66] and to the references therein for further details.

[1] In this paper $\lambda_i(\Omega)$ $(i \geqslant 1)$ stands for the ith eigenvalue of $-\Delta$ with Neumann boundary conditions on $\partial\Omega$.