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Regularity of solutions to a parabolic free boundary problem with variable coefficients

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ABSTRACT

In this work the optimal Lipschitz regularity of viscosity solutions for a type of parabolic free boundary problem with variable coefficients is proved under the main assumption that the free boundary is Lipschitz.

1. Introduction

Free boundary problems occur in many science and engineering problems when a law changes discontinuously as it crosses from one region to another. Typical problems require the solution to satisfy different sets of conditions on its positivity and negativity sets. Since these regions depend on the solution u itself, the interface between them, called the free boundary, is not known at the outset.

In this paper, we study the regularity of viscosity solutions to a family of parabolic free boundary problems of the form

$$\begin{cases} \mathcal{L}u - u_t = 0 & \text{in } (\{u > 0\} \cup \{u \le 0\}^\circ) \subset \Omega \\ G(u_\nu^+, u_\nu^-) = 1 & \text{along } \partial\{u > 0\} \subset \Omega. \end{cases} \tag{1.1}$$

Here $\mathcal{L}$ is a non-divergence form uniformly elliptic operator and $u_\nu^\pm$ denote, respectively, the inner normal derivative relative to the sets $\{u > 0\}$ and $\{u \le 0\}^\circ$.

The main goal of this paper is to prove that, under suitable conditions on G (stated precisely in the next section), viscosity solutions to (1.1) which possess a Lipschitz free boundary and satisfy a non-degeneracy condition are Lipschitz continuous. This is the optimal regularity for this problem.

Motivating examples of the free boundary condition $G(u_\nu^+, u_\nu^-)$ include $u_\nu^+ = 1$ or $(u_\nu^+)^2 - (u_\nu^-)^2 = 2M > 0$, both of which arise from a singular perturbation problem used to model a problem from combustion theory.

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This problem consists of studying the limit as $\varepsilon \to 0$ of weak solutions to

$$\Delta u^\varepsilon - u_t^\varepsilon = \beta_\varepsilon(u^\varepsilon)$$

where $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(s/\varepsilon)$ for a Lipschitz function β supported on $[0,1]$ with β_ε satisfying for some positive constants C and M

$$0 \leq \beta_\varepsilon \leq \frac{C}{\varepsilon}\chi_{(0,\varepsilon)} \quad \text{and} \quad \int_0^\varepsilon \beta_\varepsilon(s)\,\mathrm{d}s = M.$$

The one-phase version of this problem (i.e. $u^\varepsilon \geq 0$) was studied in [9]. It was shown there that the limit function u satisfies $u_\nu^+ = 1$ along the boundary of its positivity set. For two phase problem (studied in [6,7]) the boundary condition for the limit u is $(u_\nu^+)^2 - (u_\nu^-)^2 = 2M > 0$. In all of these works the boundary condition was shown to hold in a weak sense, with pointwise equality only at regular points on the zero set. This two phase boundary condition is the prototype for the function G.

An elliptic version of (1.1) with the Laplacian was studied in [4,5]. It is in these works that the main ideas used in this paper, such as monotonicity cones, viscosity solutions to (1.1), and 'sup-convolutions' were first developed. Similar methods were later applied to the study of the Stefan problem for the heat equation in [1–3]. All of these works, including the singular perturbation problem, involved only the case where $\mathcal{L} = \Delta, \Delta$ denoting the Laplacian. In [12] these results were adapted to the study of the constant coefficient version of (1.1).

When $\mathcal{L} = \Delta$, whether in the elliptic or parabolic case, extensive use is made of the fact that directional derivatives of solutions to a constant coefficient linear PDE are themselves solutions to the same PDE. In particular, tools like the Harnack Inequality can be applied to the directional derivatives. At variance with the case when $\mathcal{L} = \Delta$, directional derivatives of solutions to the operator $\mathcal{L} - \partial_t$ are not themselves solutions. This prevents a straightforward extension of the constant coefficient results to this variable coefficient case. Indeed, the only results extending these methods to a variable coefficient parabolic problem are the recent papers [10,11] for the Stefan problem.

The strategy of this work is to begin by deducing that a cone of ε-monotonicity (defined below) exists for solutions of (1.1). This is due to u being $\mathcal{L}$-caloric on a Lipschitz domain and vanishing on a portion of its boundary. We then construct an iteration which simultaneously decreases both ε and the aperture of the ε-monotonicity cone opening. By carefully balancing decrease in ε, which is desirable, with the decrease in the cone opening, which is not, we show that there does indeed exist a cone of full monotonicity up to the free boundary of u. Using this information, we deduce control of the time derivative by the spacial gradient $|\nabla u|$, then proceed to show the boundedness of $|\nabla u|$, thereby establishing Lipschitz continuity of the solution.

The structure of this work is as follows: Section 2 states precisely the problem under consideration as well as our main result. Section 3 collects the main tools and known results used in the analysis of this problem. Section 4 contains our results on the asymptotic behavior of solutions near the free boundary. These results are used in Section 5 to prove that a space–time cone of monotonicity exists up to the free boundary for u. In Section 6, we use this to prove the Lipschitz regularity of u.

2. Definitions and statement of results

We will denote the positivity set of u by Ω^+, i.e. $\Omega^+ = \{x \in \Omega \mid u(x) > 0)\}$; likewise the negative set is denoted by Ω^-. Occasionally we will write $\Omega^\pm(u)$ to emphasize the dependence of these domains on the function u. The set $\partial\{u > 0\}$ is the free boundary and will be denoted by $FB(u)$ or just FB.

We will denote by $C_{R,T}(x_0, t_0)$ the cylinder

$$B_R'(x_0) \times (t_0 - T, t_0 + T).$$

If the center of the cylinder is the origin we will simply write $C_{R,T}$ and if $R = T$ we will write C_R.