Nonlinear Analysis 146 (2016) 74–99

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Regularity of the free boundary for a Bernoulli-type parabolic problem with variable coefficients

Thomas Backing

Purdue University, Department of Mathematics, 150 N. University Street, West Lafayette, IN 47907, United States

ARTICLE INFO

Article history:
Received 12 July 2016
Accepted 23 August 2016
Communicated by Enzo Mitidieri

Keywords:
Free boundary problems
Singular perturbation

ABSTRACT

In this work the regularity of the free boundary for a class of parabolic free boundary problems with variable coefficients is studied. Under the main hypothesis that the free boundary is Lipschitz, the main result is that the free boundary is in fact $C^{1,\alpha}$ surfaces, thereby implying that the viscosity solution is in fact classical in this case.

1. Introduction

In this paper we continue the study of a class of parabolic free boundary problems initiated in [4]. Our present goal is to establish that Lipschitz free boundaries are $C^{1,\alpha}$ surfaces in a sense that is stated precisely in Theorem 1.

The problem under consideration in this work is as follows: Let u be a viscosity solution in a domain Ω to

$$\begin{cases} \mathcal{L}u - u_t = 0 \quad \text{in } \{u > 0\} \cup \{u \le 0\}^\circ \\ G(u_\nu^+, u_\nu^-) = 1 \quad \text{along } \partial\{u > 0\}. \end{cases} \tag{1.1}$$

Here $\mathcal{L}$ is an elliptic operator with Hölder continuous coefficients and $G(\, , \,)$ defines the free boundary condition of the problem.

Typical examples of the boundary condition in (1.1) include $u_\nu^+ = 1$ and $(u_\nu^+)^2 - (u_\nu^-)^2 = 2M, M$ a positive number. Both arise as the free boundary condition for a singular perturbation problem which models combustion. This problem consists of studying the limit as $\varepsilon \to 0$ of solutions to

$$\Delta u^\varepsilon - u_t^\varepsilon = \beta_\varepsilon(u^\varepsilon)$$

E-mail address: tbacking@purdue.edu.

http://dx.doi.org/10.1016/j.na.2016.08.013

where $\beta(s)$ is a Lipschitz function supported on $[0,1]$ with

$$\int_0^1 \beta(s)\mathrm{d}s = M \quad \text{and} \quad \beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right).$$

Under the assumption that $u^\varepsilon \geq 0$, it was shown in [11] that the boundary condition for the limit function u is $u_\nu^+ = 1$. In [8,9] the two phase version of this problem was studied and the free boundary condition for the limit solution was demonstrated to be $(u_\nu^+)^2 - (u_\nu^-)^2 = 2M$. In both cases this free boundary condition holds in a suitable weak sense.

Stationary (i.e. time independent) versions of (1.1) with $\mathcal{L} = \Delta$ were studied in [5,6]. In these pioneering papers the idea of a viscosity solution to (1.1) was introduced and the key concepts of monotonicity cones and sup-convolutions were introduced. The main result of these works is that Lipschitz free boundaries are smooth, as are sufficiently 'flat' free boundaries. In this context 'flat' means that the free boundary is close to the graph of a Lipschitz function with suitably small Lipschitz constant.

The first extension of these techniques to a parabolic problem was in [1–3]. The problem studied in these works is the Stefan problem, which models melting/solidification and differs from (1.1) in its free boundary condition which, among other differences, involves the time derivative of u. Similar, though not quite as strong, results were proved in these papers as for the elliptic problem studied in [5,6]. Lipschitz free boundaries were proved to be C^1 under a non-degeneracy condition on u, and sufficiently flat free boundaries were also proved to be C^1. In both cases it was also proved that $u \in C^1(\overline{\Omega^+}) \cup C^1(\overline{\Omega^-})$, so that u satisfies the free boundary condition in a classical sense. Finally, [14] adapted these techniques to the study of (1.1) for the heat equation.

All of the above cited works on the regularity of the free boundary involve either the Laplacian in the stationary case or the heat equation in the parabolic case. The proofs in these papers make extensive use of the fact that directional derivatives of solutions to a constant coefficient linear PDE are themselves solutions to the same equation. Indeed, the most difficult aspect of adapting these methods to the variable coefficient case is that this fact is unavailable. The only progress in adapting these methods to problems with variable coefficients is found in [7], where the authors study an elliptic problem, and in [12,13] where the authors study the Stefan problem with flat free boundaries.

In this work we adapt these methods and use them to study the regularity of the free boundary to solutions of (1.1). Our main result is that the free boundary is a differentiable surface whose normal varies with a Hölder modulus of continuity and the free boundary condition is taken up with continuity.

The outline of this work is as follows: In Section 2 we precisely define the problem, the concept of a solution, our assumptions and our main result. In Section 3 we have collected the main tools and known results that we will need in our analysis. Section 4 deals with the interior enlargement of the monotonicity cone while Section 5 contains results that propagate a portion of this enlargement to the free boundary. Finally Section 6 contains the iteration used to prove the regularity of the free boundary in space while Section 7 contains a similar iteration used to prove the regularity in space–time.

2. Definitions and statement of results

We collect in this section the precise statement and hypotheses of our problem along with the statement of our main result.

We will denote the positivity set of u by Ω^+; likewise the negative set is denoted by Ω^-. Occasionally we will write $\Omega^\pm(u)$ to emphasize the dependence of these domains on the function u. The set $\partial\{u > 0\}$ is the free boundary and will be denoted by $FB(u)$ or just FB. In this work we will assume that the free boundary is the graph of a Lipschitz function f, that is, it consists of the set $\{(x', x_n, t) | f(x', t) = x_n\}$ with $f(0,0) = 0$. Denote by L and L_0 the Lipschitz constant of f in space and time respectively.