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Regular blocks and Conley index of isolated invariant continua in surfaces[☆]

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ABSTRACT

In this paper we study topological and dynamical features of isolated invariant continua of continuous flows defined on surfaces. We show that near an isolated invariant continuum the flow is topologically equivalent to a C^1 flow. We deduce that isolated invariant continua in surfaces have the shape of finite polyhedra. We also show the existence of *regular isolating blocks* of isolated invariant continua and we use them to compute their Conley index provided that we have some knowledge about the truncated unstable manifold. We also see that the ring structure cohomology index of an isolated invariant continuum in a surface determines its Conley index. In addition, we study the dynamics of non-saddle sets, preservation of topological and dynamical properties by continuation and we give a topological classification of isolated invariant continua which do not contain fixed points and, as a consequence, we also classify isolated minimal sets.

1. Introduction

In this paper we study topological and dynamical features of isolated invariant continua of continuous flows $\varphi : M \times \mathbb{R} \to M$ defined on surfaces. By a surface M we mean a connected 2-manifold without boundary. To avoid trivial cases, when we refer to an isolated invariant continuum K, it will be implicit that it is a proper subset of M, i.e. $\emptyset \neq K \subsetneq M$.

The paper is structured as follows. In Section 2 we show that near an isolated invariant continuum K the flow is topologically equivalent to a C^1 flow and, as a consequence, K admits a basis of neighborhoods comprised of what we call *isolating block manifolds*. The main result of this section is Theorem 7 which establishes that an isolated invariant continuum K of a flow on a surface must have the shape of a finite polyhedron. Besides, we characterize the *initial sections* of the truncated unstable manifold $W^u(K) - K$ introduced in [3]. Section 3 is devoted to prove the main results of the paper which are Theorem 12 where the existence of the so-called *regular isolating blocks* of isolated invariant continua on surfaces is established and Theorem 16 which establishes a complete classification of the possible values taken by the Conley index of K.

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In particular, it is proven that the Conley of K is the pointed homotopy type of a wedge of circumferences if K is neither an attractor nor a repeller, the pointed homotopy type of a disjoint union of a wedge of circumferences and an external point (which is the base point) if K is an attractor and the pointed homotopy of a wedge of circumferences and a closed surface if K is a repeller. Both the number of circumferences in the wedge and the corresponding genus of the surface in the case of repellers are determined by the first Betti number of K and the knowledge of an initial section of $W^u(K) - K$. The existence of regular isolating blocks plays a key role in our proof of this classification. In Section 4 we prove Theorem 19 which is a classification of the Conley index of K in terms of the ring structure of the cohomology index. Finally, Section 5 is devoted to some applications of the previous results. The main results of this section are Theorems 26 and 29. Theorem 26 studies the preservation of some topological and dynamical properties by continuation. For instance, it is proven that if $(K_\lambda)_{\lambda \in I}$ is a continuation of an attractor (resp. repeller) K_0 then, for each λ, K_λ must have a component K_λ^1 which is an attractor (resp. repeller) with the same shape of K_0. It is also proven that the property of being saddle is preserved by continuation for small values of the parameter and that if K_λ is a continuum for each λ, the property of being non-saddle is preserved if and only if the shape is preserved. On the other hand, Theorem 29 establishes that if an isolated invariant continuum in a surface does not have fixed points it must be non-saddle and either a limit cycle, a closed annulus bounded by two limit cycles or a Möbius strip bounded by a limit cycle. A nice consequence of this result is Corollary 30 which establishes that a minimal isolated invariant continuum in a surface must be either a fixed point or a limit cycle.

We shall use through the paper the standard notation and terminology in the theory of dynamical systems. By the *omega-limit* of a set $X \subset M$ we understand the set $\omega(X) = \bigcap_{t>0} \overline{X \cdot [t, \infty)}$ while the *negative omega-limit* is the set $\omega^*(X) = \bigcap_{t>0} \overline{X \cdot (-\infty, -t]}$. The *unstable manifold* of an invariant compactum K is defined as the set $W^u(K) = \{x \in M \mid \emptyset \neq \omega^*(x) \subset K\}$. Similarly the *stable manifold* $W^s(K) = \{x \in M \mid \emptyset \neq \omega(x) \subset K\}$. For us, an *attractor* is an *asymptotically stable* compactum and a *repeller* is an asymptotically stable compactum for the reverse flow.

We shall assume in the paper some knowledge of the Conley index theory of isolated invariant compacta of flows. These are compact invariant sets K which possess a so-called isolating neighborhood, that is, a compact neighborhood N such that K is the maximal invariant set in N, or setting

$$N^+ = \{x \in N : x[0, +\infty) \subset N\}; \qquad N^- = \{x \in N : x(-\infty, 0] \subset N\};$$

such that $K = N^+ \cap N^-$. We shall make use of a special type of isolating neighborhoods, the so-called isolating blocks, which have good topological properties. More precisely, an isolating block N is an isolating neighborhood such that there are compact sets $N^i, N^o \subset \partial N$, called the entrance and exit sets, satisfying

1. $\partial N = N^i \cup N^o$,
2. for every $x \in N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset M - N$ and for every $x \in N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset M - N$,
3. for every $x \in \partial N - N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset \mathring{N}$ and for every $x \in \partial N - N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset \mathring{N}$.

These blocks form a neighborhood basis of K in M. Associated to an isolating block N there are defined two continuous functions

$$t^s : N - N^+ \to [0, +\infty), \qquad t^i : N - N^- \to (-\infty, 0]$$

given by

$$t^s(x) := \sup\{t \geq 0 \mid x[0, t] \subset N\}, \qquad t^i(x) := \inf\{t \leq 0 \mid x[t, 0] \subset N\}.$$

These functions are known as the *exit time* and the *entrance time* respectively. We shall also use the notation $n^+ = N^+ \cap \partial N$ and $n^- = N^- \cap \partial N$. If the flow is differentiable, the isolating blocks can