Nonlinear Analysis 146 (2016) 161–184

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

On the classification and evolution of bifurcation curves for a one-dimensional prescribed curvature problem with nonlinearity $\exp(\frac{au}{a+u})$

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ARTICLE INFO

Article history:
Received 25 March 2016
Accepted 22 August 2016
Communicated by S. Carl

MSC:
34B18
74G35

Keywords:
Prescribed curvature problem
Exact multiplicity
Positive solution
Bifurcation curve
Time map

ABSTRACT

We study the classification and evolution of bifurcation curves of positive solutions u for the one-dimensional prescribed curvature problem

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \lambda \exp\left(\dfrac{au}{a+u}\right), & -L<x<L, \\ u(-L)=u(L)=0, \end{cases}$$

where $\lambda > 0$ is a bifurcation parameter, and $L, a > 0$ are two evolution parameters. We prove that, on $(\lambda, \|u\|_\infty)$-plane, for $0 < a \le 36/17 \approx 2.118$, the bifurcation curve is ⊃-shaped. While for $a > 36/17$, the bifurcation curve is ⊃-shaped or reversed ε-like shaped. In particular, for $a > a^{**} \approx 4.107$, the bifurcation curve is (i) ⊃-shaped if $L > 0$ small enough and (ii) reversed ε-like shaped if L is large enough.

1. Introduction

In this paper, we study the classification and evolution of bifurcation curves of positive solutions $u \in C^2(-L, L) \cap C[-L, L]$ for the one-dimensional prescribed curvature problem

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \lambda f(u), & -L<x<L, \\ u(-L)=u(L)=0, \end{cases} \tag{1.1}$$

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http://dx.doi.org/10.1016/j.na.2016.08.012

where $\lambda > 0$ is a bifurcation parameter, $L > 0$ is an evolution parameter and the nonlinearity

$$f(u) \equiv \exp\left(\frac{au}{a+u}\right), \quad a > 0. \tag{1.2}$$

This nonlinearity $f(u)$ satisfies $f(0) = 1$, $f'(u) > 0$ on $[0, \infty)$, $\lim_{u\to\infty} f(u) = \exp(a) > 0$. In addition,

1. if $a \le 2$, then f is *concave* on $(0, \infty)$;
2. if $a > 2$, then f is *convex–concave* on $(0, \infty)$. More precisely, f is convex on $(0, \gamma)$ and concave on (γ, ∞) where $\gamma = a(a - 2)/2 > 0$ is the unique inflection point of f.

The one-dimensional prescribed curvature problem

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1 + (u'(x))^2}}\right)' = \lambda \tilde{f}(u), \quad -L < x < L, \\ u(-L) = u(L) = 0, \end{cases} \tag{1.3}$$

and n-dimensional problem of it, with general nonlinearity $\tilde{f}(u)$ or with many different types nonlinearities, like $\exp(u)$, $\exp(u) - 1$, $\exp\left(\frac{au}{a+u}\right) - 1$, $(1 + u)^p$ $(p > 0)$, u^p $(p > 0)$, a^u $(a > 0)$, $u - u^3$, $(1 - u)^{-p}$ $(p > 0)$ and $u^p + u^q$ $(0 \le p < q < \infty)$ have been recently investigated by many authors, see e.g. [1,3,4,7–11,14,17,19,21–23,26,27,25,29,28,30,31,33]. Note that, in geometry, a solution $u(x)$ of (1.3) is also called a graph of prescribed curvature $\lambda \tilde{f}(u)$.

A solution $u \in C^2(-L, L) \cap C[-L, L]$ of (1.3) with $u' \in C([-L, L], [-\infty, \infty])$ is called classical if $|u'(\pm L)| < \infty$, and it is called non-classical if $u'(-L) = \infty$ or $u'(L) = -\infty$, see [21,27]. In this paper, we always allow that solutions $u \in C^2(-L, L) \cap C[-L, L]$ satisfy $u' \in C([-L, L], [-\infty, \infty])$. Notice that it can be shown that (see [4,27]), for problem (1.3),

(i) Any non-trivial solution $u \in C^2(-L, L) \cap C[-L, L]$ is concave and positive on $(-L, L)$ if $\tilde{f}(u) > 0$ for $u > 0$ since the equation in (1.3) can be written in the equivalent form

$$u''(x) = -\lambda(1 + u'^2)^{3/2} \tilde{f}(u) < 0 \quad \text{on } (-L, L).$$

(ii) A positive solution $u \in C^2(-L, L) \cap C[-L, L]$ must be symmetric on $[-L, L]$ if $\tilde{f} : [0, \infty) \to [0, \infty)$ is a continuous function satisfying $\tilde{f}(u) > 0$ for $u > 0$. Thus $u'(-L) = -u'(+L)$.

(iii) A classical solution $u \in C^2(-L, L) \cap C[-L, L]$ with $u' \in C([-L, L], [-\infty, \infty])$ belongs to $C^2[-L, L]$.

(iv) A non-classical solution $u \in C^2(-L, L) \cap C[-L, L]$ with $u' \in C([-L, L], [-\infty, \infty])$ satisfies $|u'(\pm L)| = \infty$ by symmetry of positive solutions u of (1.3) if $\tilde{f}$ is a continuous, positive function.

We define the bifurcation curve C_a of (1.1), (1.2) by

$$C_a \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1), (1.2)}\}.$$

We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve C_a is $\supset$-shaped (see e.g. Fig. 3 depicted below) if there exists $\lambda^* > 0$ such that C_a has exactly one turning point at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ where the bifurcation curve C_a turns to the left. In addition, we say that the bifurcation curve C_a is reversed ε-shaped (see e.g. Fig. 9(ii-1)–(ii-11) depicted below) if C_a has turning points at some points $(\lambda_1, \|u_{\lambda_1}\|_\infty)$, $(\lambda_2, \|u_{\lambda_2}\|_\infty)$ and $(\lambda_3, \|u_{\lambda_3}\|_\infty)$ satisfying

(i) $\lambda_1 > \lambda_2$ and $\lambda_3 > \lambda_2$;

(ii) $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty < \|u_{\lambda_3}\|_\infty$;