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Viscosity solutions of second order partial differential equations with boundary conditions on Riemannian manifolds

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ABSTRACT

The comparison, uniqueness and existence for viscosity solution to a class of fully nonlinear second order partial differential equations with boundary condition on a finite dimensional Riemannian manifold are studied. Some results regarding the regularity of the boundary are obtained. These results show that every compact manifold with $C^{1,1}$ boundary satisfies the regularity condition.

1. Introduction

Viscosity solutions were introduced by M. G. Crandall and P. L. Lions [9] on Euclidean spaces in the 1980s. In the following years the theory has been considerably expanded. One of the most significant achievement has been the treatment of second order degenerate elliptic equations at the end of the 1980s. The primary advantages of the theory of viscosity solutions to the fully nonlinear second order partial differential equations are that it provides very general existence and uniqueness theorems by allowing merely continuous functions to be solutions and that it yields precise formulations of general boundary conditions; see, e.g., [6,8] and the references therein.

Hamilton–Jacobi equations arise naturally in the setting of Riemannian manifolds, see [1]. In [4] by introducing a theory of first order nonsmooth calculus, the existence and uniqueness of viscosity solutions to Hamilton–Jacobi equations on Riemannian manifolds were studied. In [5] the comparison, uniqueness and existence of viscosity solutions to second order PDEs of the form $F(x, u, du, d^2u) = 0$ are derived, where $u : M \to \mathbb{R}$ and M is a complete Riemannian manifold. Then the boundary condition $u = f$ on $\partial\Omega$, where Ω is an open subset of M, is studied along with the same equation with no boundary condition on M.

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In this paper we study the viscosity solutions to second order partial differential equations with boundary condition of the general form

$$(\mathrm{BVP})\begin{cases}(\mathrm{E})\ F(x,u,du,d^2u)=0, & \text{on } \Omega;\\ (\mathrm{BC})\ B(x,u,du)=0, & \text{on } \partial\Omega,\end{cases}$$

where $u:\overline{\Omega}\to\mathbb{R}$, Ω is an open subset of a finite dimensional Riemannian manifold M, $\partial\Omega$ and $\overline{\Omega}$ are its boundary and closure, respectively. We also present a characterization of exterior ball condition in terms of geometric properties of the boundary $\partial\Omega$. By using this characterization, we obtain a sufficient condition for the regularity of the boundary to prove the uniqueness of solutions of (BVP).

There are simple PDEs with Neumann boundary condition which do not admit smooth solutions all the time, even in $\mathbb{R}^n$, (see [2]) such as

$$\begin{cases}-\Delta u=g(x,u(x)), & \text{on } \Omega;\\ \dfrac{\partial u}{\partial n}=f(x,u(x)), & \text{on } \partial\Omega.\end{cases}$$

where f and g are sufficiently regular functions and n is the unit outward pointing normal to $\partial\Omega$. These types of questions arise naturally on Riemannian manifolds in the study of conformal geometry [3,18], KMGP systems [11], eigenvalues of Neumann problems, etc. Motivated by this observation, we consider their solutions in the viscosity sense. In fact we show that under some general assumptions on F and $\overline{\Omega}$ the comparison holds for Neumann and Dirichlet problems on Riemannian Manifolds. In particular, we prove that in the case of Neumann problem there exists a unique solution.

Let us briefly explain what we do in this paper. In Section 2 we consider the following condition

($\natural$): for every $z\in\partial\Omega$ there exist a convex neighborhood W of z and $r>0$ such that for every $x\in W\cap\partial\Omega$ and $y\in W\cap\overline{\Omega}$ we have

$$\langle n(x),\exp_x^{-1}y\rangle\le\frac{1}{r}\|\exp_x^{-1}y\|^2.$$

By considering the concept of uniform exterior sphere condition (UESC), we show that ($\natural$) is equivalent to UESC. This result has interesting geometrical meaning specially in the theory of sets with positive reach or prox-regular sets in Riemannian manifolds. We conclude this section by showing that every $C^{1,1}$ compact manifolds with boundary satisfies ($\natural$).

In Section 3 we consider second order viscosity subdifferentials (superjets and subjets) of a function defined on a subset S of a Riemannian manifold M. Indeed, by using the definition of second order subjet (see [5]) of a lower semicontinuous function $\tilde{f}:M\to[-\infty,+\infty)$, we define the subjet $J_S^{2,-}f(x)$ of a lower semicontinuous function $f:S\to[-\infty,+\infty)$. This notion enables us to consider the viscosity subsolutions and supersolutions of (BVP). An upper semicontinuous function $u:\overline{\Omega}\to\mathbb{R}$ is called a viscosity subsolution of $F=0$ (or a viscosity solution of $F\le 0$) on Ω if for every $x\in\Omega$

$$F(x,u(x),\xi,X)\le 0\quad\text{for every } (\xi,X)\in J_{\Omega}^{2,+}u(x).$$

It seems reasonable to define subsolution of (BVP) to be an upper semicontinuous function on $\overline{\Omega}$ such that it is a viscosity subsolution of $F=0$ on Ω and $B=0$ on $\partial\Omega$ simultaneously. This notion of solution which is called strong solution does not preserve stability. This observation suggests a more appropriate definition of solutions of (BVP) (see [14]). This definition is presented in this section.

Sections 4 and 5 are devoted to establish comparison between subsolution u and supersolution v of (BVP). As in [5,8], the well known maximum principle plays a natural role in our approach. In particular,