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Global existence to the initial–boundary value problem for a system of semilinear wave equations

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A B S T R A C T

We prove the global existence of finite energy and H_2 solutions for a system of nonlinear wave equations in a bounded domain. One equation of the system has a dissipative term, while another equation has no dissipative mechanism. To derive the delicate a priori estimates we employ a 'loan' method.

1. Introduction

In this paper we first consider the initial–boundary value problem for a system of nonlinear wave equations of the form:

$$u_{tt} - \Delta u + \rho(u_t) = f(u,v) \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$v_{tt} - \Delta v = g(u,v) \quad \text{in } \Omega \times (0,\infty), \tag{1.2}$$

with the initial–boundary conditions

$$\begin{aligned} &u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x) \\ &\text{and} \quad u(x,t)|_{\partial\Omega} = v(x,t)|_{\partial\Omega} = 0, \end{aligned} \tag{1.3}$$

where Ω is a bounded domain in $R^N, 1 \leq N \leq 4$, with a C^1 class boundary $\partial\Omega$. The nonlinear dissipative term $\rho(u_t)$ is a function like $\rho(u_t) = |u_t|^r u_t, 0 \leq r \leq 2/(N-2)^+$, and the functions $f(u,v), g(u,v)$ denote nonlinear source terms.

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The existence of global solutions for nonlinear wave equations with source type nonlinearity was first studied by Sattinger [23], where the famous potential well method was developed. Related ideas were applied to nonlinear wave equations with nonlinear dissipative terms by several authors (cf. [13], Nakao and Narazaki [20], Ikehata, Matsuyama and Nakao [9] and Ikehata [8], etc.). But, such a method cannot be applied to systems and there seem to be very few results for a system like (1.1)–(1.2). Alves, Cavalcanti and others [1] considered a system of nonlinear wave equations, but, in [1] both equations have nonlinear dissipative terms and also the source terms are of special form such as $f(u,v) = F_u(u,v)$ and $g(u,v) = F_v(u,v)$. For such a special case we can apply a potential well method.

When the growth orders of nonlinear dissipative terms are greater in some senses than those of the nonlinear source terms the existence problem of global solutions has been discussed by many authors for systems as well as single equations (cf. Georgiev and Todorova [7], Serrin, Todorova and Vitillaro [24], M. Rammaha and Strei [22], Barbu, Lasiecka and Rammaha [3], Agre and Rammaha [2], and Rammaha and Sakuntasathien [21], etc.). But, we treat the case where the order of nonlinear dissipation is rather less than those of the source terms, in particular, the second equation of our system does not include any dissipative term, and it seems that the methods in these papers cannot be applied to our problem.

In the present paper we assume that

$$|f(u,v)| \le k|u|^\alpha |v|^\beta, \quad \alpha \ge 1,\ \beta \ge 0$$

and

$$|g(u,v)| \le k|u|^p |v|^q, \quad p,q \ge 0.$$

Then we can expect that for small initial data, the energy of the first equation

$$E_u(t) \equiv \frac{1}{2}\left(\|u_t(t)\|^2 + \|\nabla u(t)\|^2\right)$$

would decay at the rate $t^{-2/r}$ as $t \to \infty$ under some additional conditions on the exponents α and β for a small initial data, and consequently the energy of the second equation

$$E_v(t) \equiv \frac{1}{2}\left(\|v_t(t)\|^2 + \|\nabla v(t)\|^2\right)$$

would remain bounded under appropriate conditions on α, β, p, q and r. These a priori estimates would imply the global existence of finite energy solutions. In this paper we first show that such a conjecture in fact holds if $1 \le N \le 3$. The argument would become easier if the second equation also has a dissipative term like $\tilde{\rho}(v_t) = |v_t|^m v_t$, and the case without dissipation like (1.2) is more interesting.

We make the following hypotheses on $f(u,v)$, $g(u,v)$ and $\rho(w)$.

Hyp.*A*. $f(u,v)$ and $g(u,v)$ are continuous functions on R^2 and satisfy:

(1)

$$|f(u,v)| \le k_1|u|^\alpha |v|^\beta, \qquad |g(u,v)| \le k_1|u|^p|v|^q$$

with $\alpha \ge 1, \beta \ge 0, p \ge 0$ and $q \ge 0$ for some $k_1 > 0$,

(2)

$$|f(u_1,v_1) - f(u_2,v_2)| \le k_1(|u_1| + |u_2| + |v_1| + |v_2| + 1)^m(|u_1 - u_2| + |v_1 - v_2|)$$

and