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A model for the quasistatic growth of cracks with fractional dimension☆

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ABSTRACT

We study a variational model for the quasistatic growth of cracks with fractional dimension in brittle materials. We give a minimal set of properties of the collection of admissible cracks which ensure the existence of a quasistatic evolution. Both the antiplane and the planar cases are treated.

1. Introduction

The current models of linearly elastic fracture mechanics are based on Griffith's energy criterion [12]. Crack growth is the competition between the elastic energy released when the crack grows and the energy spent to produce new crack. In the standard case of planar elasticity it is tacitly assumed that cracks open along one-dimensional sets K in the reference configuration $\Omega \subset \mathbb{R}^2$, so that it is natural to assume that the energy spent to produce a crack K is proportional to its one-dimensional Hausdorff measure $\mathcal{H}^1(K)$. The elastic energy stored in the uncracked region is given by

$$\mathcal{E}_{\mathbb{C}}(u,K) := \frac{1}{2}\int_{\Omega\setminus K} \mathbb{C}(x)Eu(x) : Eu(x)\,\mathrm{d}x,$$

where $\mathbb{C}(x)$ is the elasticity tensor, Eu is the symmetrized gradient of the displacement $u\colon \Omega\setminus K \to \mathbb{R}^2$, and: denotes the scalar product of 2×2 matrices.

☆ This paper is dedicated to Nicola Fusco on the occasion of his 60th birthday.

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In [11] Francfort and Marigo introduced a quasistatic variational model based on these ideas (we refer the reader to [5] for a general review on the variational approach to fracture mechanics). A time-dependent boundary datum $w(t,x)$ is prescribed on a sufficiently regular portion $\partial_D\Omega$ of the boundary $\partial\Omega$. In the discrete-time formulation, for every integer $n > 1$ one divides the time interval $[0,T]$ by $n+1$ subdivision points $0 = t_n^0 < t_n^1 < \cdots < t_n^n = T$. Assuming for simplicity that the proportionality constant in the crack energy is equal to 1, the solution (u_n^i, K_n^i) at time t_n^i is obtained inductively by minimizing the functional

$$\mathcal{E}_{\mathbb{C}}^{\mathrm{tot}}(u,K) := \mathcal{E}_{\mathbb{C}}(u,K) + \mathcal{H}^1(K), \tag{1.1}$$

among all pairs (u,K) such that $K \supset K_n^{i-1}$, u is sufficiently smooth in $\Omega \setminus K$, and $u = w(t_n^i)$ on $\partial_D\Omega \setminus K$.

To implement this minimization scheme it is important to fix the class $\mathcal{K}$ of admissible cracks K that we consider and to make precise the notion of smoothness for u on $\Omega \setminus K$ (see, *e.g.*, Definitions 2.2 and 6.1). The existence of solutions to these problems has been obtained in [10] for the antiplane case and in [6] for the planar case, assuming that $\mathcal{K}$ is the set of all compact subsets of $\overline{\Omega}$ with an *a priori* uniform bound on the number of connected components. This assumption is crucial to obtain the lower semicontinuity of $\mathcal{H}^1(K)$ with respect to the Hausdorff distance (see Definition 2.1).

In many brittle materials the assumption that $\mathcal{H}^1(K) < +\infty$, though useful from the mathematical point of view, is not physically justified, see [1–4,17,20].

Our aim in this paper is to extend the results in [6,10] to the case where the collection $\mathcal{K}$ of admissible cracks is composed of α-dimensional sets, for some $\alpha \in (1,2)$. We shall see that in this case we cannot take as $\mathcal{K}$ the collection of all compact subsets of $\overline{\Omega}$ with a uniform bound on the number of connected components. The purpose of this paper is to determine which further properties of $\mathcal{K}$ are needed to obtain the results for α-dimensional cracks.

In the discrete-time formulation, the minimum problem (1.1) is replaced by the minimum problem for

$$\mathcal{E}_{\mathbb{C}}^{\mathrm{tot}}(u,K) := \mathcal{E}_{\mathbb{C}}(u,K) + \mathcal{H}^\alpha(K) \tag{1.2}$$

among all pairs (u,K) such that $K \in \mathcal{K}$, $K \supset K_n^{i-1}$, u is sufficiently smooth in $\Omega \setminus K$, and $u = w(t_n^i)$ on $\partial_D\Omega \setminus K$.

The first difficulty is that $\mathcal{H}^\alpha$ is not lower semicontinuous with respect to the Hausdorff metric for $\alpha \in (1,2)$, even though $\mathcal{K}$ is the collection of all the connected compact subsets of $\overline{\Omega}$. An immediate counterexample is given by the approximation of an α-dimensional connected fractal set K by polygonal pre-fractals K_n: in fact, $\mathcal{H}^\alpha(K_n) = 0$ for all n, while $\mathcal{H}^\alpha(K) > 0$.

To solve the minimum problem for the energy (1.2), we assume that the collection $\mathcal{K}$ of admissible cracks satisfies the following properties:

(1) (compactness) $\mathcal{K}$ is compact for the Hausdorff metric;
(2) (lower semicontinuity) $\mathcal{H}^\alpha$ is lower semicontinuous on $\mathcal{K}$ with respect to the Hausdorff metric.

These properties allow us to tackle the minimum problem (1.2) by the direct method of the calculus of variations (see Proposition 5.1).

To prove the existence of a quasistatic evolution based on the incremental minimum problems for (1.2), we introduce the piecewise constant interpolants $(u_n(t), K_n(t))$ defined by

$$u_n(t) := u_n^i, \qquad K_n(t) := K_n^i, \quad \text{for } t \in [t_n^i, t_n^{i+1}).$$

As in [10] we can apply a version of Helly's Theorem and obtain, for a subsequence independent of t and not relabeled, that $K_n(t)$ converges in the Hausdorff metric to a set $K(t) \in \mathcal{K}$. Moreover, there exists a function $u(t)$ such that, for a suitable subsequence (possibly depending on t), the functions $Eu_n(t)$ converge to $Eu(t)$