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Rigidity and stability of Caffarelli's log-concave perturbation theorem

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To Nicola Fusco, for his 60th birthday, con affetto e ammirazione

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ABSTRACT

In this note we establish some rigidity and stability results for Caffarelli's log-concave perturbation theorem. As an application we show that if a 1-log-concave measure has almost the same Poincaré constant as the Gaussian measure, then it almost splits off a Gaussian factor.

1. Introduction

Let γ_n denote the centered Gaussian measure in $\mathbb{R}^n$, i.e., $\gamma_n = (2\pi)^{-n/2}e^{-|x|^2/2}dx$, and let μ be a probability measure on $\mathbb{R}^n$. By a classical theorem of Brenier [2], there exists a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $T = \nabla\varphi : \mathbb{R}^n \to \mathbb{R}^n$ transports γ_n onto μ, i.e., $T_\sharp\gamma_n = \mu$, or equivalently

$$\int h \circ T\, d\gamma_n = \int h\, d\mu \quad \text{for all continuous and bounded functions } h \in C_b(\mathbb{R}^n)\ .$$

In the sequel we will refer to T as the *Brenier map* from γ_n to μ.

In [4,5] Caffarelli proved that if μ is "more log-concave" than γ_n, then T is 1-Lipschitz, that is, all the eigenvalues of $D^2\varphi$ are bounded from above by 1. Here is the exact statement:

Theorem 1.1 (*Caffarelli*)**.** *Let γ_n be the Gaussian measure in $\mathbb{R}^n$, and let $\mu = e^{-V}dx$ be a probability measure satisfying $D^2V \geq \mathrm{Id}_n$. Consider the Brenier map $T = \nabla\varphi$ from γ_n to μ. Then T is* 1-Lipschitz. *Equivalently, $0 \leq D^2\varphi(x) \leq \mathrm{Id}_n$ for a.e. x.*

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This theorem allows one to show that optimal constants in several functional inequalities are extremized by the Gaussian measure. More precisely, let F, G, H, L, J be continuous functions on $\mathbb{R}$ and assume that F, G, H, J are nonnegative, and that H and J are increasing. For $\ell \in \mathbb{R}_+$ let

$$\lambda(\mu,\ell) := \inf\left\{ \frac{H\Big(\int J(|\nabla u|)\, d\mu\Big)}{F\Big(\int G(u)\, d\mu\Big)} : u \in \mathrm{Lip}(\mathbb{R}^n), \int L(u)\, d\mu = \ell \right\}. \tag{1.1}$$

Then

$$\lambda(\gamma_n,\ell) \le \lambda(\mu,\ell). \tag{1.2}$$

Indeed, given a function u admissible in the variational formulation for μ, we set $v := u \circ T$ and note that, since $T_\sharp \gamma_n = \mu$,

$$\int K(v)\, d\gamma_n = \int K(u\circ T)\, d\gamma_n = \int K(u)\, d\mu \quad \text{for } K = G, L.$$

In particular, this implies that v is admissible in the variational formulation for γ_n. Also, thanks to Caffarelli's Theorem,

$$|\nabla v| \le |\nabla u| \circ T\, |\nabla T| \le |\nabla u| \circ T,$$

therefore

$$H\Big(\int J(|\nabla v|)\, d\gamma_n\Big) \le H\Big(\int J(|\nabla u|)\circ T\, d\gamma_n\Big) = H\Big(\int J(|\nabla u|)\, d\mu\Big).$$

Thanks to these formulas, (1.2) follows easily.

Note that the classical Poincaré and Log-Sobolev inequalities fall in the above general framework. For instance, choosing $H(t) = F(t) = L(t) = t$, $\ell = 0$, and $J(t) = F(t) = |t|^p$ with $p \ge 1$, we deduce that

$$\inf\left\{ \frac{\int |\nabla u|^p\, d\mu}{\int |u|^p\, d\mu} \,:\, u \in \mathrm{Lip}(\mathbb{R}^n), \int u\, d\mu = 0 \right\} \ge \inf\left\{ \frac{\int |\nabla u|^p\, d\gamma_n}{\int |u|^p\, d\gamma_n} \,:\, u \in \mathrm{Lip}(\mathbb{R}^n), \int u\, d\gamma_n = 0 \right\}. \tag{1.3}$$

Two questions that naturally arise from the above considerations are:

– *Rigidity*: What can be said about μ when $\lambda(\mu,\ell) = \lambda(\gamma_n,\ell)$?
– *Stability*: What can be said about μ when $\lambda(\mu,\ell) \approx \lambda(\gamma_n,\ell)$?

Looking at the above proof, these two questions can usually be reduced to the study of the corresponding ones concerning the optimal map T in Theorem 1.1 (here $|A|$ denotes the operator norm of a matrix A):

– *Rigidity*: What can be said about μ when $|\nabla T(x)| = 1$ for a.e. x?
– *Stability*: What can be said about μ when $|\nabla T(x)| \approx 1$ (in suitable sense)?

Our first main result states that if $|\nabla T(x)| = 1$ for a.e. x then μ "splits off" a Gaussian factor. More precisely, it splits off as many Gaussian factors as the number of eigenvalues of $\nabla T = D^2\varphi$ that are equal to 1. In the following statement and in the sequel, given $p \in \mathbb{R}^k$ we denote by $\gamma_{p,k}$ the Gaussian measure in $\mathbb{R}^k$ with barycenter p, that is, $\gamma_{p,k} = (2\pi)^{-k/2} e^{-|x-p|^2/2} dx$.

Theorem 1.2 (*Rigidity*)**.** *Let γ_n be the Gaussian measure in $\mathbb{R}^n$, and let $\mu = e^{-V} dx$ be a probability measure with $D^2 V \ge \mathrm{Id}_n$. Consider the Brenier map $T = \nabla\varphi$ from γ_n to μ, and let*

$$0 \le \lambda_1(D^2\varphi(x)) \le \cdots \le \lambda_n(D^2\varphi(x)) \le 1$$