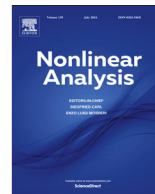




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Rigidity and stability of Caffarelli's log-concave perturbation theorem

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ABSTRACT

In this note we establish some rigidity and stability results for Caffarelli's log-concave perturbation theorem. As an application we show that if a 1-log-concave measure has almost the same Poincaré constant as the Gaussian measure, then it almost splits off a Gaussian factor.

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1. Introduction

Let γ_n denote the centered Gaussian measure in \mathbb{R}^n , i.e., $\gamma_n = (2\pi)^{-n/2} e^{-|x|^2/2} dx$, and let μ be a probability measure on \mathbb{R}^n . By a classical theorem of Brenier [2], there exists a convex function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $T = \nabla\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ transports γ_n onto μ , i.e., $T_{\#}\gamma_n = \mu$, or equivalently

$$\int h \circ T d\gamma_n = \int h d\mu \quad \text{for all continuous and bounded functions } h \in C_b(\mathbb{R}^n).$$

In the sequel we will refer to T as the *Brenier map* from γ_n to μ .

In [4,5] Caffarelli proved that if μ is “more log-concave” than γ_n , then T is 1-Lipschitz, that is, all the eigenvalues of $D^2\varphi$ are bounded from above by 1. Here is the exact statement:

Theorem 1.1 (Caffarelli). *Let γ_n be the Gaussian measure in \mathbb{R}^n , and let $\mu = e^{-V} dx$ be a probability measure satisfying $D^2V \geq \text{Id}_n$. Consider the Brenier map $T = \nabla\varphi$ from γ_n to μ . Then T is 1-Lipschitz. Equivalently, $0 \leq D^2\varphi(x) \leq \text{Id}_n$ for a.e. x .*

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This theorem allows one to show that optimal constants in several functional inequalities are extremized by the Gaussian measure. More precisely, let F, G, H, L, J be continuous functions on \mathbb{R} and assume that F, G, H, J are nonnegative, and that H and J are increasing. For $\ell \in \mathbb{R}_+$ let

$$\lambda(\mu, \ell) := \inf \left\{ \frac{H\left(\int J(|\nabla u|) d\mu\right)}{F\left(\int G(u) d\mu\right)} : u \in \text{Lip}(\mathbb{R}^n), \int L(u) d\mu = \ell \right\}. \tag{1.1}$$

Then

$$\lambda(\gamma_n, \ell) \leq \lambda(\mu, \ell). \tag{1.2}$$

Indeed, given a function u admissible in the variational formulation for μ , we set $v := u \circ T$ and note that, since $T_{\#}\gamma_n = \mu$,

$$\int K(v) d\gamma_n = \int K(u \circ T) d\gamma_n = \int K(u) d\mu \quad \text{for } K = G, L.$$

In particular, this implies that v is admissible in the variational formulation for γ_n . Also, thanks to Caffarelli's Theorem,

$$|\nabla v| \leq |\nabla u| \circ T |\nabla T| \leq |\nabla u| \circ T,$$

therefore

$$H\left(\int J(|\nabla v|) d\gamma_n\right) \leq H\left(\int J(|\nabla u|) \circ T d\gamma_n\right) = H\left(\int J(|\nabla u|) d\mu\right).$$

Thanks to these formulas, (1.2) follows easily.

Note that the classical Poincaré and Log-Sobolev inequalities fall in the above general framework. For instance, choosing $H(t) = F(t) = L(t) = t$, $\ell = 0$, and $J(t) = F(t) = |t|^p$ with $p \geq 1$, we deduce that

$$\inf \left\{ \frac{\int |\nabla u|^p d\mu}{\int |u|^p d\mu} : u \in \text{Lip}(\mathbb{R}^n), \int u d\mu = 0 \right\} \geq \inf \left\{ \frac{\int |\nabla u|^p d\gamma_n}{\int |u|^p d\gamma_n} : u \in \text{Lip}(\mathbb{R}^n), \int u d\gamma_n = 0 \right\}. \tag{1.3}$$

Two questions that naturally arise from the above considerations are:

- *Rigidity*: What can be said about μ when $\lambda(\mu, \ell) = \lambda(\gamma_n, \ell)$?
- *Stability*: What can be said about μ when $\lambda(\mu, \ell) \approx \lambda(\gamma_n, \ell)$?

Looking at the above proof, these two questions can usually be reduced to the study of the corresponding ones concerning the optimal map T in Theorem 1.1 (here $|A|$ denotes the operator norm of a matrix A):

- *Rigidity*: What can be said about μ when $|\nabla T(x)| = 1$ for a.e. x ?
- *Stability*: What can be said about μ when $|\nabla T(x)| \approx 1$ (in suitable sense)?

Our first main result states that if $|\nabla T(x)| = 1$ for a.e. x then μ “splits off” a Gaussian factor. More precisely, it splits off as many Gaussian factors as the number of eigenvalues of $\nabla T = D^2\varphi$ that are equal to 1. In the following statement and in the sequel, given $p \in \mathbb{R}^k$ we denote by $\gamma_{p,k}$ the Gaussian measure in \mathbb{R}^k with barycenter p , that is, $\gamma_{p,k} = (2\pi)^{-k/2} e^{-|x-p|^2/2} dx$.

Theorem 1.2 (Rigidity). *Let γ_n be the Gaussian measure in \mathbb{R}^n , and let $\mu = e^{-V} dx$ be a probability measure with $D^2V \geq \text{Id}_n$. Consider the Brenier map $T = \nabla\varphi$ from γ_n to μ , and let*

$$0 \leq \lambda_1(D^2\varphi(x)) \leq \dots \leq \lambda_n(D^2\varphi(x)) \leq 1$$

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