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Remark on a nonlocal isoperimetric problem

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ABSTRACT

We consider isoperimetric problem with a nonlocal repulsive term given by the Newtonian potential. We prove that regular critical sets of the functional are analytic. This optimal regularity holds also for critical sets of the Ohta–Kawasaki functional. We also prove that when the strength of the nonlocal part is small the ball is the only possible stable critical set.

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1. Introduction

In this short note we study critical sets of the functional

$$J(E) = P(E) + \gamma \int_E \int_E G(x,y)\,dxdy \tag{1}$$

where $G(\cdot,\cdot)$ is the standard Newtonian kernel

$$G(x,y) = \begin{cases} \dfrac{1}{2\pi} \log\left(\dfrac{1}{|x-y|}\right) & (n=2) \\ \dfrac{1}{n(n-2)\omega_n} \dfrac{1}{|x-y|^{n-2}} & (n \geq 3) \end{cases}$$

and $P(E)$ denotes the surface measure, or the perimeter of the set E. This model was first introduced by Gamow [19] in the physically relevant case $n = 3$ to model the stability of atomic nuclei. It also rises as a

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ground state problem from the Ohta–Kawasaki functional introduced by Ohta and Kawasaki [34] to model diblock copolymers. In the periodic setting the Ohta–Kawasaki functional can be written as

$$J_{\mathbb{T}^n}(E) = P(E) + \gamma \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} G_{\mathbb{T}^n}(x,y)u_E(x)u_E(y)dxdy, \tag{2}$$

where $G_{\mathbb{T}^n}(\cdot,\cdot)$ is the Green's function in the flat torus and $u_E = 2\chi_E - 1$. Both with (1) and (2) we are interested in minimizing the functional under volume constraint.

There has been an increasing interest among mathematicians to study the above functionals [1,2,6,9–11, 17,21–23,27,28,31,37,40]. Besides from the obvious physical applications, the main motivation to study (1) and (2) is that they feature the competition between a short range interfacial force, described here by the perimeter, which prefers the minimizer to be smooth and connected and a long range repulsive force which prefers the minimizer to be scattered. Indeed, under volume constraint the ball minimizes the perimeter by the isoperimetric inequality, while it maximizes the nonlocal part (see e.g. [30]).

By a scaling argument we notice that when the volume is small the nonlocal term in (1) becomes small. This suggest that for $n \geq 3$ the ball should be the minimizer of (1) under volume constraint when the volume is small, or equivalently when γ is small. This was first proved in [25,28] and generalized in [6] to more general potentials and in [16] to nonlocal perimeter. On the other hand when the volume is large the repulsive term becomes stronger and it was proved in [28,31] for $n = 3$ that the minimization problem does not have a solution.

In this note we are interested in critical sets which are not necessarily minimizers. The first result of this paper concerns the regularity of critical sets. To state the result we denote the Newtonian potential by

$$v_E(x) = \int_E G(x,y)dy. \tag{3}$$

The Euler equation associated with (1) can be then written as

$$H_E + 2\gamma v_E = \lambda \quad \text{on } \partial E, \tag{4}$$

where H_E is the mean curvature. We say that a C^2-regular set (the boundary is a C^2-hypersurface) is critical if it satisfies (4). It was proved in [1,11,28] that regular critical sets are $C^{3,\alpha}$-regular for every $\alpha \in (0,1)$ and then in [26] that they are in fact C^∞-regular. We use the method developed in [29] to prove the sharp regularity of critical sets.

Theorem 1. *If $E \subset \mathbb{R}^n$ is a regular critical set of (1) then it is analytic.*

We note that the above result holds also for critical sets of the Ohta–Kawasaki functional (2). For the minimizers of (1) and (2) we obtain that they are analytic up to a singular set whose Hausdorff dimension is at most $n-8$. Theorem 1 can also be applied to improve the partial regularity result for general non-smooth critical sets in [24].

Our second result concerns the uniqueness of stable critical sets. The study of critical sets is mathematically interesting for two reasons. First, it is closely related to the stability of the Alexandroff theorem on sets of constant mean curvatures, since in the case $\gamma = 0$ we know by Alexandroff theorem that the only connected critical sets are balls. Second, if we do not have any constraint on γ in (1) then the family of possible critical sets is much richer than the family of minimizers. Indeed, it easy to see that for large enough γ an annulus is a critical set (see [21]). More interesting examples of critical sets which are diffeomorphic to the torus are constructed in [35].

In the planar case the functional (1) does not have a minimizer due to the logarithmic behavior of the potential. However, it is showed in [21] that in the plane when γ is small the disk is the unique critical set of (1). We would like to have a similar result in higher dimension, but this turns out to be challenging

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