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Quasilinear elliptic systems with measure data

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ABSTRACT

We study the existence of solutions of quasilinear elliptic systems involving N equations and a measure on the right hand side, with the form

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{\beta=1}^{N}\sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u)\frac{\partial}{\partial x_j}u^\beta\right) = \mu^\alpha & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha \in \{1,\ldots,N\}$ is the equation index, Ω is an open bounded subset of $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}^N$ and μ is a finite Randon measure on $\mathbb{R}^n$ with values in $\mathbb{R}^N$. Existence of a solution is proved for two different sets of assumptions on A. Examples are provided that satisfy our conditions, but do not satisfy conditions required on previous works on this matter.

1. Introduction

Let us consider the Dirichlet elliptic problem

$$-\text{div}\,[A(x,u(x),Du(x))] = \mu \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, μ is a measure on $\mathbb{R}^n$ with values into $\mathbb{R}^N$ and A satisfies suitable coercivity and growth conditions. We note that (1.1) is a system of N equations.

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First consider the case $N = 1$, i.e. (1.1) is only one single equation. Existence of distributional solutions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ has been deeply studied, starting from [7], see [9,11,8,32,4] and the survey [5]. Uniqueness seems to be a delicate matter, e.g. see [33,3,19] and the introduction of [12]. Regularity results are contained in [28–30,10,22,2] and the survey [31] (see also [6]). Note that existence of solutions is usually obtained by a truncation argument, which shows why the vectorial case $N \geq 2$ is difficult and only few contributions are available in the literature. In fact, for systems $N \geq 2$, the p-Laplacian $A(x, y, \xi) = |\xi|^{p-2}\xi$ is treated in [18,13], and the anisotropic case, in which each component of the gradient $D_i u$ may have a possibly different exponent p_i, is dealt in [23,24]. Let us write (1.1) using components, that is,

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[A_i^\alpha(x, u(x), Du(x))\right] = \mu^\alpha \quad \text{for } \alpha \in \{1, \dots, N\}. \tag{1.3}$$

We note that systems more general than the p-Laplacian are considered in [14,16], under the assumption

$$0 \leq \sum_{\alpha=1}^{N} \sum_{i=1}^{n} A_i^\alpha(x, y, \xi)((Id - b \times b)\xi)_i^\alpha \tag{1.4}$$

for every $b \in \mathbb{R}^N$ with $|b| \leq 1$. In [34], the author assumes the componentwise sign condition

$$0 \leq \sum_{i=1}^{n} A_i^\alpha(x, y, \xi)\xi_i^\alpha \tag{1.5}$$

for every $\alpha \in \{1, \dots, N\}$. When $N = 2$, (1.4) implies (1.5), since it is enough to take first $b = (1, 0)$ and then $b = (0, 1)$. In [25], the authors consider that A is independent of y and satisfies the componentwise coercivity condition

$$\nu|\xi^\alpha|^2 - M \leq \sum_{i=1}^{n} A_i^\alpha(x, \xi)\xi_i^\alpha \tag{1.6}$$

for every $\alpha \in \{1, \dots, N\}$, for some constants $\nu \in (0, +\infty)$ and $M \in [0, +\infty)$. In [15], they relax (1.4) to some extent

$$-c|\xi|^q - g(x) \leq \sum_{\alpha=1}^{N} \sum_{i=1}^{n} A_i^\alpha(x, y, \xi)((Id - b \times b)\xi)_i^\alpha \tag{1.7}$$

for some $c \in [0, +\infty)$, $g \in L^1(\Omega)$ and $q \in [1, n)$ where $\xi \mapsto A(x, y, \xi)$ is n-coercive.

We note that in [19,17,26] the authors do not truncate u; they modify Du and then adjust via Hodge decomposition; such a procedure requires the dimension n to be the exponent in the coercivity condition for A. The authors use nice estimates for Hodge decomposition, which have been studied in [20] (see also appendix A, in [21]).

In the present paper we consider quasilinear systems, i.e. systems (1.3) with

$$A_i^\alpha(x, y, \xi) = \sum_{\beta=1}^{N} \sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x, y)\, \xi_j^\beta, \tag{QL}$$

where the coefficients $a_{i,j}^{\alpha,\beta}(x, y)$ are measurable with respect to x and continuous with respect to y. Moreover, we assume ellipticity for the diagonal coefficients $a_{i,j}^{\alpha,\alpha}$, the off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$ (with $\alpha \neq \beta$) are sufficiently small, and all the coefficients are bounded. We prove existence of distributional solutions to (1.1)–(1.2) under two sets of hypotheses, with different assumptions on the off-diagonal coefficients.

The first result deals with off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$ (with $\alpha \neq \beta$) that have support contained in a "*staircase*" set, along the diagonals of the $y^\alpha - y^\beta$ plane (see assumption (A_5) in Section 2 and Fig. 1).

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