ARTICLE IN PRESS

Nonlinear Analysis ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

On formulae decoupling the total variation of BV functions

Augusto C. Ponce[a], Daniel Spector[b,*]

[a] Université catholique de Louvain, Institut de Recherche en Mathématique et Physique, Chemin du cyclotron 2, L7.01.02, 1348 Louvain-la-Neuve, Belgium

[b] National Chiao Tung University, Department of Applied Mathematics, Hsinchu, Taiwan

ARTICLE INFO

Article history:
Received 29 March 2016
Accepted 31 August 2016
Communicated by Enzo Mitidieri

To Nicola Fusco, a master of BV, on the occasion of his sixtieth birthday

Keywords:
Bounded variation
Fractional Laplacian
Non-local energies

ABSTRACT

In this paper we prove several formulae that enable one to capture the singular portion of the measure derivative of a function of bounded variation as a limit of non-local functionals. One special case shows that rescalings of the fractional Laplacian of a function $u \in SBV$ converge strictly to the singular portion of Du.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded and smooth subset (or all of $\mathbb{R}^N$) and u be a function of bounded variation in Ω, i.e. $u \in L^1(\Omega)$ and its distributional derivative Du is a Radon measure with finite total variation

$$|Du|(\Omega) := \sup_{\substack{\Phi \in C_c^1(\Omega;\mathbb{R}^N),\\ \|\Phi\|_{L^\infty(\Omega;\mathbb{R}^N)} \leq 1}} \int_\Omega u \operatorname{div} \Phi. \tag{1.1}$$

The space of such functions $BV(\Omega)$ contains the Sobolev space $W^{1,1}(\Omega)$, with strict inclusion, since in general for $u \in BV(\Omega)$ one has the decomposition of the measure

$$Du = \nabla u \, \mathcal{L}^N + D^s u,$$

where ∇u is the Radon–Nikodym density of Du with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$ and $D^s u$ is singular with respect to $\mathcal{L}^N$. In particular, $u \in W^{1,1}(\Omega)$ precisely when $D^s u \equiv 0$.

* Corresponding author.
E-mail addresses: Augusto.Ponce@uclouvain.be (A.C. Ponce), dspector@math.nctu.edu.tw (D. Spector).

http://dx.doi.org/10.1016/j.na.2016.08.028

Please cite this article in press as: A.C. Ponce, D. Spector, On formulae decoupling the total variation of BV functions, Nonlinear Analysis (2016), http://dx.doi.org/10.1016/j.na.2016.08.028

While the computation of the total variation (1.1) through the theory of distributions is classical, in recent years there has been an interest in its – and other related energies – approximation through the asymptotics of non-local functionals [1,4–7,9,11,12,17,14,18,20,22,24,23,25,26]. For example, Bourgain, Brezis and Mironescu [7] had shown that for $u \in W^{1,p}(\Omega), 1 \leq p < +\infty$, one has

$$\lim_{\alpha \to 1^-} (1-\alpha) \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+\alpha p}} \,\mathrm{d}y \,\mathrm{d}x = C_{p,N} \int_\Omega |\nabla u|^p. \tag{1.2}$$

Here, for $p = 1$,

$$C_{1,N} = \int_{\mathbb{S}^{N-1}} |e \cdot v| \,\mathrm{d}\mathcal{H}^{N-1}(v) = 2\omega_{N-1},$$

where $e \in \mathbb{R}^N$ is any unit vector and ω_{N-1} is the volume of the unit ball in $\mathbb{R}^{N-1}$. Formula (1.2) expresses the fact that appropriately scaled Gagliardo semi-norms tend to the total variation as the differential parameter tends to one. Their result includes the case $u \in W^{1,1}(\Omega)$, while for general $u \in BV(\Omega)$ it was proved by Dávila [15, Theorem 1] that

$$\lim_{\alpha \to 1^-} (1-\alpha) \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|^{N+\alpha}} \,\mathrm{d}y \,\mathrm{d}x = C_{1,N} |Du|(\Omega). \tag{1.3}$$

The Gagliardo semi-norms can be thought heuristically as the energy of derivatives of fractional order $\alpha \in (0,1)$, while the space of p-integrable functions for which they are finite coincides with the $W^{\alpha,p}$ space obtained by *real interpolation* of L^p and $W^{1,p}$ with parameter α. Indeed, it was subsequently shown by Milman [24] that the convergence (1.2) can be alternatively deduced from the theory of interpolation.

In the *complex interpolation* between L^p and $W^{1,p}$ when $\Omega = \mathbb{R}^N$, one classically sees the fractional Laplacian

$$(-\Delta)^{\alpha/2} u(x) := c_\alpha \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+\alpha}} \,\mathrm{d}y$$

filling the role of the differential object of order $\alpha \in (0,1)$, and even in a more extended range, though with a different formula. The constant

$$c_\alpha = \frac{2^{\alpha-1} \alpha \Gamma\left(\frac{N+\alpha}{2}\right)}{\pi^{\frac{N}{2}} \Gamma\left(\frac{2-\alpha}{2}\right)}$$

ensures that the Fourier transform of the fractional Laplacian satisfy $\widehat{(-\Delta)^{\alpha/2}} u(\xi) = (2\pi|\xi|)^\alpha \widehat{u}(\xi)$.

As integer powers of the Laplacian can also be understood from the framework of the functional calculus, the natural limits here are $\alpha = 0$ and $\alpha = 2$. This observation is supported by the trivial convergence, as α tends to 1, of the energies

$$\lim_{\alpha \to 1^-} (1-\alpha) \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+\alpha}} \,\mathrm{d}y \right|^p \mathrm{d}x = 0, \tag{1.4}$$

whenever $u \in W^{1,p}(\mathbb{R}^N)$ and $p \geq 1$. The case $p > 1$ follows from the L^p boundedness of the Riesz transforms that yields the equivalence between the quantities $\|\nabla u\|_{L^p(\mathbb{R}^N)}$ and $\|(-\Delta)^{1/2} u\|_{L^p(\mathbb{R}^N)}$; see e.g. [19, Lemma 3.6]. The case $p = 1$ requires a different argument that can be found in [27,28], based on the approximation of u by smooth functions. Alternatively, one can still obtain the expected energy, in the spirit of (1.2), via a non-local gradient approach [23].

What is perhaps surprising is that the convergence (1.4) is no longer true for general $u \in BV(\Omega)$ and $p = 1$. For example, in the case where $u = \chi_A$ is the characteristic function of a set of finite perimeter, Dávila's formula (1.3) and the straightforward integral identity

$$\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \frac{\chi_A(x)-\chi_A(y)}{|x-y|^{N+\alpha}} \,\mathrm{d}y \right| \mathrm{d}x = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\chi_A(x)-\chi_A(y)|}{|x-y|^{N+\alpha}} \,\mathrm{d}y \,\mathrm{d}x, \tag{1.5}$$

Please cite this article in press as: A.C. Ponce, D. Spector, On formulae decoupling the total variation of BV functions, Nonlinear Analysis (2016), http://dx.doi.org/10.1016/j.na.2016.08.028