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Traveling solutions and evolution properties of the higher order Camassa–Holm equation

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ABSTRACT

This paper investigates general mild traveling solutions and the stationary solution of the higher-order Camassa–Holm equation. By argument of regularity and conservation, we construct admitted traveling solutions of the second order and the third order Camassa–Holm equations. Furthermore, evolution procedure and evolution properties of a traveling solution are discussed; the discussion reveals the periodic behavior of the traveling solution curve itself and its curvature. These investigations are valuable to understand forming and evolution of a pattern.

1. Introduction

In 1993, a new evolution equation was rediscovered by Camassa and Holm in [2] as a model of shallow water waves as KdV equation does before. The equation is usually termed Camassa–Holm equation (CH equation) although it first appeared in the works of Fuchssteiner and Fokas [16]. Its new properties such as completely integrability, bi-Hamiltonian structure and peaked solitary solutions have aroused people's intense attention and interest on generic equations of shallow water waves. Many researches have shown that there are many similar properties between the CH equation and the KdV equation; for instance, both the KdV equation and the CH equation are approximate models describing geodesic flow [12]. However, there are also significant differences; especially, the solitary solution of the KdV equation is smooth but that of the CH equation is peak.

The higher-order Camassa–Holm equation describes exponential curves of the manifold of smooth orientation preserving diffeomorphisms of the unit circle in the plane. When they investigated geodesic flow on the group Diff (S^1), Constantin and Kolev obtained a new equation in [10] as follows:

$$\partial_t u = B_k(u, u), \tag{1.1}$$

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where

$$B_k(u,u) = A_k^{-1} C_k(u,u) - u\partial_x u,$$
$$A_k(u) = \sum_{j=0}^{k} (-1)^j \partial_x^{2j} u,$$
$$C_k(u) = -uA_k(\partial_x u) + A_k(u\partial_x u) - 2\partial_x u A_k(u).$$

Actually, the Cauchy problem of Eq. (1.1) was first studied by Coclite, Holden and Karlsen in [5].

Letting $k=0$, Eq. (1.1) yields

$$u_t + 3uu_x = 0, \tag{1.2}$$

which is the famous Burgers equation without viscosity. Letting $k=1$, Eq. (1.1) is changed to the standard CH equation

$$\partial_t u - \partial_t \partial_x^2 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0. \tag{1.3}$$

For $k \geq 2$ Eq. (1.1) is called higher-order Camassa–Holm equation, and the second order Camassa–Holm equation is

$$\partial_t u - \partial_t \partial_x^2 u + \partial_t \partial_x^4 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u + 2\partial_x u \partial_x^4 u + u\partial_x^5 u = 0. \tag{1.4}$$

It is well known that there are many works on the CH equation (1.3), so we do not mention those works here and readers can refer to [8,11,20,9,4,18,1,17, and references therein]. There are also several studies about the well-posedness of the Cauchy problem for the higher-order Camassa–Holm equation. In [5], Coclite, Holden and Karlsen established the existence of global weak solutions and proved a "weak equals strong" uniqueness result. They claimed the following: Suppose $u_0 \in H_{k,p} =: \left\{ f \in H^k(R) \,\middle|\, \partial^k f \in L^p(R) \right\}$, where $2 < p < +\infty$, then there exists a unique global solution $u \in C\left(\left([0,+\infty); C^{k-1}(R)\right) \cap L^\infty\left([0,+\infty); H^k(R)\right)\right)$ for the Cauchy problem of Eq. (1.1) with initial value $u_0 = u(0,x)$. Furthermore, for any $T > 0, u \in L^\infty\left([0,T]; H^{k+1}(R)\right)$ provided $u_0 \in H^{k+1}(R)$; and $u \in L^\infty([0,T]; H_{k,r}), 2 \leq r < +\infty$, provided $u_0 \in H_{k,r}$. In [15], Ding and Lv obtained the result that suppose $u_0 \in H^k(R)$ and constant $M > 0$ such that $P_{>M} u_0 = 0$, then there exists a unique global conservative solution $u \in C\left([0,+\infty); H^{k-1}(R)\right) \cap L^\infty\left([0,+\infty); H^k(R)\right)$ for Cauchy problem of the higher-order Camassa–Holm equation (1.1) with initial data u_0. In [13], Ding and Liu also proved the local well-posedness for the Cauchy problem of Eq. (1.1). For the case $k=2$, the global existence for the Cauchy problem of Eq. (1.1) was investigated by Tian, Zhang and Xia in [19]. In [3], Coclite and Ruvo considered the convergence of the solution of the high order Camassa–Holm equation.

On view of the structure, the difference between the KdV equation, the CH equation and the higher-order Camassa–Holm equation mainly relies on the operator $A_k = \sum_{j=0}^{k} (-1)^j \partial_x^{2j}$. Obviously, A_k is a linear differential operator. So there are some similar properties between the KdV equation and the CH equation and the higher-order Camassa–Holm equation. However, the terms on which the linear differential operator A_k act are nonlinear. It may be the nonlinearity that arouses those differences between the CH equation and the higher-order Camassa–Holm equation.

This paper focuses on traveling solutions of the higher-order Camassa–Holm equation (1.1) and their evolution procedure and evolution properties. Both the smooth solitary solution of the KdV equation and the peaked solution of the CH equation decay uniformly as $t \to \infty$. Although traveling solutions constructed in this paper for the higher-order equation (1.1) decay as $t \to \infty$, the procedure is not uniform. The investigation on the evolution procedure of the solution is helpful to understand the forming and evolution of a pattern. On the other hand, the traveling wave solutions of the greatest height of the governing equations for water waves have a peak at their crest [6,7], The discussion of crests is valuable to comprehend the discussion in this paper.

The remainder of the paper is organized as follows: Section 2 discusses stationary solution and generality mild traveling solutions of Eq. (1.1); Section 3 constructs admitted traveling solutions of the second and