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# Existence of positive solutions for nonlinear elliptic equations with convection terms

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ABSTRACT

We prove the existence of positive solutions for the equation

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,u,\nabla u)=f(x,u,\nabla u)\quad \text{in } \Omega$$

under the Dirichlet boundary condition, where the essential point is the dependence of the terms of the elliptic equation on the solution $u$ and its gradient $\nabla u$. We develop an approach based on approximate solutions and on a new strong maximum principle.



## 1. Introduction

In this paper, we study the existence of positive solutions for the following nonlinear elliptic equation

$$\begin{cases} -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,u,\nabla u)=f(x,u,\nabla u) & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases} \tag{P}$$

on a bounded domain $\Omega\subset\mathbb{R}^N$ with $C^2$ boundary $\partial\Omega$.

Throughout the paper we assume that the lower order term $f(x,u,\nabla u)$ fulfills the condition:

(F) $f$ is a continuous function on $\overline{\Omega}\times[0,\infty)\times\mathbb{R}^N$ satisfying

$$f(x,0,\xi)\geq 0$$

for every $(x,\xi)\in\Omega\times\mathbb{R}^N$ as well as

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(f1) there exist constants $p > 1, r_1 > 1$ and $b_1 > 0$ such that

$$|f(x,t,\xi)| \le b_1(1 + t^{r_1-1} + |\xi|^{p+1}) \tag{1}$$

for all $(x,t,\xi) \in \Omega \times [0,\infty) \times \mathbb{R}^N$;

(f2) there exist constants $b_2 > 0$ and $1 < r_2 < p$ such that

$$f(x,t,\xi)t \le b_2(1 + t^{r_2} + |\xi|^p) \tag{2}$$

for all $(x,t,\xi) \in \Omega \times [0,\infty) \times \mathbb{R}^N$.

For example, the following function $f$ verifies (f1) and (f2):

$$f(x,t,\xi) = t^{r_3-1}(2 - |\xi|^{p_1}) + \frac{|\xi|^{p_2}}{1+t^{r_4}} - t^{r_5-1} + |\xi|^{p_3} \tag{3}$$

with $0 \le p_1 < p+1, 1 \le r_3 < p, 0 \le p_2 \le p, r_4 \ge 1, r_5 > 1$ and $0 \le p_3 < p-1$.

Setting

$$A(x,t,\xi) := [a_i(x,t,\xi)]_i = (\, a_1(x,t,\xi), \ldots, a_N(x,t,\xi)\,),$$

our Eq. (P) reads as

$$\begin{cases} -\mathrm{div}\, A(x,u,\nabla u) = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

We say that $u \in W_0^{1,p}(\Omega)$ is a (weak) solution of (P) if it holds

$$\int_\Omega A(x,u,\nabla u)\nabla\varphi\, dx = \int_\Omega f(x,u,\nabla u)\varphi\, dx \tag{4}$$

for all $\varphi \in W_0^{1,p}(\Omega)$ provided the integrals exist.

The generality of problem (P) is two-fold:

(i) the left-hand side of the equation is expressed in divergence form through a possibly non-homogeneous operator depending on the solution $u$ and its gradient $\nabla u$;

(ii) the nonlinearity in the right-hand side of the equation is a so-called convection term, which means that it depends on the solution $u$ and its gradient $\nabla u$.

It is for the first time when such a general problem is considered. Moreover, the growth allowed in (f1) and (f2) improves the corresponding conditions in the preceding works. We briefly describe some related results. The existence of positive solutions for problem (P) with

$$A(x,u,\nabla u) = |\nabla u|^{p-2}\nabla u,$$

i.e., $\mathrm{div}\, A(x,u,\nabla u)$ stands for the $p$-Laplacian $\Delta_p$, and with some convection term $f(x,u,\nabla u)$ is studied in [4,13,17]. Specifically, in [4] the focus is on the case of $f(x,u,\nabla u)$ with $(p-1)$-sublinear growth in $u$ and $\nabla u$. In [13] it is assumed the growth

$$\max\{0, |u|^{r_1} - C|\nabla u|^{r_2}\} \le f(x,u,\nabla u)| \le C(1 + |u|^{r_1} + |\nabla u|^{r_2})$$

with

$$p-1 < r_1 < \frac{N(p-1)}{N-p}, \qquad p-1 < r_2 < \frac{pr_1}{r_1+1}, \quad p < N.$$