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Large time behavior of solutions to the compressible Navier–Stokes equations around periodic steady states

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ABSTRACT

This paper shows that the strong solution to the compressible Navier–Stokes equation around spatially periodic stationary solution in a periodic layer of $\mathbb{R}^n (n = 2, 3)$ exists globally in time if Reynolds and Mach numbers are sufficiently small. It is proved that the asymptotic leading part of the perturbation is given by a solution to the one-dimensional viscous Burgers equation multiplied by a spatially periodic function when $n = 2$, and by a solution to the two-dimensional heat equation multiplied by a spatially periodic function when $n = 3$.

1. Introduction

This paper studies the stability of stationary solutions to the compressible Navier–Stokes equation

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \tag{1.1}$$

$$\rho(\partial_t v + (v \cdot \nabla)v) - \mu\Delta v - (\mu + \mu')\nabla \mathrm{div}\, v + \nabla p(\rho) = \rho g \tag{1.2}$$

in a periodic layer Ω_* of $\mathbb{R}^n$ with $n = 2, 3$:

$$\Omega_* = \{x = (x', x_n); x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1},\ \omega_{*,1}(x') < x_n < \omega_{*,2}(x')\}.$$

Here $\omega_{*,1}$ and $\omega_{*,2}$ are smooth Q_*-periodic functions in x' with the periodic cell $Q_* = \prod_{j=1}^{n-1}[-\frac{\pi}{\alpha_{*,j}}, \frac{\pi}{\alpha_{*,j}})$ for constants $\alpha_{*,j} > 0$ $(j = 1, \dots, n-1)$, namely, $\omega_{*,1}$ and $\omega_{*,2}$ are smooth functions satisfying $\omega_{*,j}(x' + \frac{2\pi}{\alpha_i}\mathbf{e}'_i) = \omega_{*,j}(x')$ $(i = 1, \dots, n-1, j = 1, 2)$ with $\mathbf{e}'_i = {}^\top(0, \dots, \overset{i}{1}, \dots, 0) \in \mathbb{R}^{n-1}$; $\rho = \rho(x,t)$ and $v = {}^\top(v^1(x,t), \dots, v^n(x,t))$ denote the unknown density and velocity, respectively, at $x \in \Omega_*$ and $t \geq 0$; $p = p(\rho)$ is the pressure that is assumed to be a smooth function of ρ satisfying

$$p'(\rho_*) > 0$$

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for a given constant $\rho_* > 0$; $g = {}^\top(g^1(x), \dots, g^n(x))$ is a given external force. Here and in what follows the superscript ${}^\top\cdot$ stands for the transposition. μ and μ' are the viscosity coefficients that are assumed to be constants satisfying $\mu > 0$, $\frac{2}{n}\mu + \mu' \geq 0$. We assume that $\frac{\mu'}{\mu}$ satisfies

$$\frac{\mu'}{\mu} \leq \mu_1 \tag{1.3}$$

for a certain constant $\mu_1 > 0$. We also assume that $g = {}^\top(g^1(x), \dots, g^n(x))$ is Q_*-periodic in x'.

In [8,29], it was shown that if g is sufficiently small, then the system (1.1)–(1.2) under the boundary condition

$$v|_{x_n=\omega_{*,j}(x')} = 0 \quad (x' \in \mathbb{R}^{n-1},\ j = 1, 2) \tag{1.4}$$

has a Q_*-periodic stationary solution $\overline{u}_s = {}^\top(\overline{\rho}_s, \overline{v}_s)$ whose components $\overline{\rho}_s$ and $\overline{v}_s$ are in general non-uniform in x' and x_n.

The purpose of this paper is to prove the nonlinear stability of $\overline{u}_s$ when the Reynolds and Mach numbers are sufficiently small.

The global existence of strong solutions of the multi-dimensional compressible Navier–Stokes equations was proved by Matsumura and Nishida [22–24] around the motionless state $u_s = {}^\top(\rho_*, 0)$ when the underlying domain is the whole space, half space and exterior domains. After their pioneering works, large time behavior of solutions around the motionless state in unbounded domains has been investigated in detail. See, e.g.,
[7,9,13,17–19,21,22,24,27] for the cases of multi-dimensional whole space, half space and exterior domains. On the other hand, the study of behavior of solutions around stationary solution $u_s = {}^\top(\rho_s, v_s)$ with non-uniform velocity v_s is still under development. A difficulty in the mathematical analysis appears due to the non-uniform velocity field of stationary flows which makes the hyperbolic aspect of Eqs. (1.1)–(1.2) stronger, and thus, the stability analysis is getting more difficult compared with that of the motionless state. As for the stability of stationary flow with non-uniform velocity field on the whole space, we mention that Shibata and Tanaka [26,27] proved the existence and the stability of stationary solutions for small external forces and established the decay rate of perturbations. See also [16,20,28] for the stability of time-periodic solutions on the whole space. Kagei and others [10,15] studied the stability of parallel flows in a flat layer which are simple flows with non-uniform velocities. It was proved that parallel flow is asymptotically stable for small initial perturbations if the Reynolds and Mach numbers are sufficiently small, and a detailed asymptotic description of the large time was established. (See also [2,4,3] for parallel flows in a cylindrical domain, and [5,6] for time-periodic parallel flows.)

In this paper we extend the stability analysis for parallel flows to the one for stationary flows $\overline{u}_s = {}^\top(\overline{\rho}_s, \overline{v}_s)$ with non-uniform spatially periodic velocity fields $\overline{v}_s$. In [8], spectral properties of the linearized semigroup around $\overline{u}_s = {}^\top(\overline{\rho}_s, \overline{v}_s)$ was investigated. (See also [14].) Based on the results in [8] we study the nonlinear problem in this paper. We show that the large time behavior of the perturbation is described by a solution of a one-dimensional viscous Burgers equation in the case of $n = 2$ and by a two-dimensional linear heat equation in the case of $n = 3$ provided that the Reynolds and Mach numbers are sufficiently small.

We briefly state the main result of this paper. After introducing suitable non-dimensional variables, the equations for the perturbation $u = {}^\top(\phi, w) = {}^\top(\gamma^2(\rho - \rho_s), v - v_s)$ take the following form:

$$\partial_t \phi + \mathrm{div}(\phi v_s) + \gamma^2 \mathrm{div}(\rho_s w) = f^0, \tag{1.5}$$

$$\partial_t w - \frac{\nu}{\rho_s}\Delta w - \frac{\tilde{\nu}}{\rho_s}\nabla \mathrm{div}\, w + \nabla\Big(\frac{P'(\rho_s)}{\gamma^2 \rho_s}\phi\Big) + \frac{1}{\gamma^2 \rho_s^2}(\nu \Delta v_s + \tilde{\nu}\nabla \mathrm{div}\, v_s)\phi + v_s \cdot \nabla w + w \cdot \nabla v_s = \tilde{f}, \tag{1.6}$$

$$w|_{\partial\Omega} = 0, \tag{1.7}$$

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \tag{1.8}$$