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On the periodic and asymptotically periodic nonlinear Helmholtz equation

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ABSTRACT

In the first part of this paper, the existence of infinitely many L^p-standing wave solutions for the nonlinear Helmholtz equation

$$-\Delta u - \lambda u = Q(x) \mid u \mid^{p-2} u \quad \text{in } \mathbb{R}^N$$

is proven for $N \geq 2$ and $\lambda > 0$, under the assumption that Q be a nonnegative, periodic and bounded function and the exponent p lies in the Helmholtz subcritical range. In a second part, the existence of a nontrivial solution is shown in the case where the coefficient Q is only asymptotically periodic.

1. Introduction

In this paper, we consider for $N \geq 2$ the semilinear equation

$$-\Delta u - \lambda u = Q(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1}$$

where $\lambda > 0$, Q is a bounded and nonnegative function, and the exponent p lies in the subcritical range $2_* < p < 2^*$, where

$$2_* := \frac{2(N+1)}{N-1} \quad \text{and} \quad 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geq 3 \\ \infty & \text{if } N = 2. \end{cases}$$

We study existence of real-valued solutions $u: \mathbb{R}^N \to \mathbb{R}$ that decay to zero at infinity. Such solutions correspond, via the Ansatz

$$\psi(t,x) := e^{i\sqrt{\lambda}t}u(x),$$

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to weakly spatially decaying, standing wave solutions of the Nonlinear Wave Equation

$$\partial_{tt}\psi - \Delta\psi = Q(x)|\psi|^2\psi, \quad t \in \mathbb{R},\ x \in \mathbb{R}^N.$$

Under general assumptions on Q, the problem (1) cannot be handled using direct variational methods, since its solutions (if any) are not expected to decay faster than $O(|x|^{-\frac{N-1}{2}})$ at infinity and will therefore not belong to the space $L^2(\mathbb{R}^N)$. Recently, a dual method has been set up, which allows to study this problem variationally. Using it, the existence of nontrivial solutions lying in $L^q(\mathbb{R}^N)$ for all $q \geq p$ and admitting an expansion of the form

$$\lim_{R\to\infty} \frac{1}{R}\int_{B_R} \left| u(x) + 2\left(\frac{2\pi}{\sqrt{\lambda}|x|}\right)^{\frac{N-1}{2}} \operatorname{Re}\left[e^{i\sqrt{\lambda}|x| - \frac{i(N-1)\pi}{4}} g_u\left(\frac{x}{|x|}\right)\right]\right|^2 dx = 0,$$

where $g_u(\xi) = -\frac{i}{4}\left(\frac{\lambda}{2\pi}\right)^{\frac{N-2}{2}} \mathcal{F}\left(Q|u|^{p-2}u\right)(\sqrt{\lambda}\xi)$, $\xi \in S^{N-1}$, was proven (see [16,17]). Here, $\mathcal{F}$ denotes the Fourier transform. More precisely, infinitely many nontrivial bounded $W^{2,p}(\mathbb{R}^N)$-solutions were obtained under the assumption $\lim_{|x|\to\infty} Q(x) = 0$ (see also [15] where more general nonlinearities were considered). For periodic Q, only the existence of a nontrivial solution (pair) was proven, and one of the main goals of the present paper is to show that (1) in fact possesses infinitely many geometrically distinct solutions in $W^{2,p}(\mathbb{R}^N)$.

In the case where $\lambda < 0$, or more generally when $-\lambda$ is replaced by a bounded and periodic function V satisfying $\inf V > 0$, results giving the existence of infinitely many solutions for (1) go back to the work of Coti Zelati and Rabinowitz [12]. Shortly after, Alama and Li [5] using a dual method, and Kryszewski and Szulkin [19] using a linking argument and a new degree theory, extended this result to the case where 0 lies in a spectral gap of the Schrödinger operator $-\Delta + V(x)$. More recently, multiplicity results were given for the periodic Schrödinger equation with more general nonlinearities (see e.g. [1,2,11,13,22,24] and the references therein).

In the present paper, we show that the proof scheme developed by Szulkin and Weth in [24, Theorem 1.2] for the periodic Schrödinger equation and used recently by Squassina and Szulkin [22] in the case of a logarithmic nonlinearity, can be adapted to work in combination with the dual variational method. In particular, using the framework of [17], we obtain the following result.

Theorem 1.1. *Let $2_* < p < 2^*$ and consider $Q \in L^\infty(\mathbb{R}^N)\setminus\{0\}$ nonnegative and $\mathbb{Z}^N$-periodic. Then (1) has infinitely many geometrically distinct pairs of strong solutions $\pm u_n \in W^{2,q}(\mathbb{R}^N)$, $p \leq q < \infty$.*

The proof (see Section 3) is based on the fact that, assuming that only finitely many geometrically distinct solutions exist, one is able to show the discreteness of Palais–Smale sequences for the energy functional and then apply a deformation argument to get a contradiction. The main technical point, when passing to the dual method is to find suitable estimates to prove the analogue of [24, Lemma 2.14]. This is done in Lemma 3.2 using the reverse Hölder inequality and the recently proven nonvanishing property for the subcritical Helmholtz equation [17, Theorem 3.1] for $N \geq 3$ and [16, Theorem 3.1] for $N = 2$.

Among one of the problems most extensively studied in the context of nonlinear Schrödinger equation is the case of asymptotically periodic or asymptotically autonomous nonlinearity. Starting with the work by Ding and Ni [14] based on Lions' concentration-compactness principle, many authors studied this case (see e.g. [7,10,21] and the references therein). In Section 4 of the present paper, we give an analogous existence result for the nonlinear Helmholtz equation. We namely study (1) in the case where the coefficient Q satisfies $|Q(x) - Q_\infty(x)| \to 0$ as $|x| \to \infty$ for some periodic function Q_∞. Under assumptions close to those of Ding and Ni [14], we obtain the existence of a nontrivial solution (see Theorem 4.3). Our proof uses the fibering method applied to the dual energy functional, and we show that the dual ground state is attained under these assumptions.