Nonlinear Analysis 152 (2017) 116–148

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Well-posedness, global existence and large time behavior for Hardy–Hénon parabolic equations

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ARTICLE INFO

Article history:
Received 22 August 2016
Accepted 14 December 2016
Communicated by Enzo Mitidieri

MSC:
primary 35B40
35B30
35K58
35K67
35K57
secondary 35B33

Keywords:
Hardy–Hénon parabolic equation
Well-posedness
Large time behavior

ABSTRACT

In this paper we study the nonlinear parabolic equation $\partial_t u = \Delta u + a|x|^{-\gamma}|u|^\alpha u$, $t > 0$, $x \in \mathbb{R}^N \setminus \{0\}$, $N \geq 1$, $a \in \mathbb{R}$, $\alpha > 0$, $0 < \gamma < \min(2, N)$ and with initial value $u(0) = \varphi$. We establish local well-posedness in $L^q(\mathbb{R}^N)$ and in $C_0(\mathbb{R}^N)$. In particular, the value $q = N\alpha/(2-\gamma)$ plays a critical role.

For $\alpha > (2-\gamma)/N$, we show the existence of global self-similar solutions with initial values $\varphi(x) = \omega(x)|x|^{-(2-\gamma)/\alpha}$, where $\omega \in L^\infty(\mathbb{R}^N)$ is homogeneous of degree 0 and $\|\omega\|_\infty$ is sufficiently small. We then prove that if $\varphi(x) \sim \omega(x)|x|^{-(2-\gamma)/\alpha}$ for $|x|$ large, then the solution is global and is asymptotic in the L^∞-norm to a self-similar solution of the nonlinear equation. While if $\varphi(x) \sim \omega(x)|x|^{-\sigma}$ for $|x|$ large with $(2-\gamma)/\alpha < \sigma < N$, then the solution is global but is asymptotic in the L^∞-norm to $e^{t\Delta}(\omega(x)|x|^{-\sigma})$.

The equation with more general potential, $\partial_t u = \Delta u + V(x)|u|^\alpha u$, $V(x)|x|^\gamma \in L^\infty(\mathbb{R}^N)$, is also studied. In particular, for initial data $\varphi(x) \sim \omega(x)|x|^{-(2-\gamma)/\alpha}$, $|x|$ large, we show that the large time behavior is linear if V is compactly supported near the origin, while it is nonlinear if V is compactly supported near infinity.
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1. Introduction

In this paper we consider the singular nonlinear parabolic equation

$$\partial_t u = \Delta u + a|\cdot|^{-\gamma}|u|^\alpha u, \tag{1.1}$$

$u = u(t,x) \in \mathbb{R}$, $t > 0$, $x \in \mathbb{R}^N \setminus \{0\}$, $N \geq 1$, $a \in \mathbb{R}$, $\alpha > 0$, $\gamma > 0$ and with initial value

$$u(0) = \varphi. \tag{1.2}$$

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http://dx.doi.org/10.1016/j.na.2016.12.008
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The case $\gamma = 0$ corresponds to the standard nonlinear heat equation. For $\gamma < 0$ it is known in the literature as a Hénon parabolic equation, while if $\gamma > 0$ it is known as a Hardy parabolic equation. In this paper we are concerned with the case $\gamma > 0$. We are interested in the well-posedness of (1.1) with initial data $\varphi \in L^q(\mathbb{R}^N)$, $1 \le q < \infty$, and in $C_0(\mathbb{R}^N)$. We also study the existence of global solutions, including self-similar solutions and prove the existence of asymptotically self-similar solutions.

In what follows, we denote $\|.\|_{L^q(\mathbb{R}^N)}$ by $\|.\|_q$, $1 \le q \le \infty$. For all $t > 0$, $e^{t\Delta}$ denotes the heat semi-group

$$\left(e^{t\Delta} f\right)(x) = \int_{\mathbb{R}^N} G(t, x-y) f(y) dy, \tag{1.3}$$

where

$$G(t,x) = (4\pi t)^{-N/2} e^{-\frac{|x|^2}{4t}}, \quad t > 0,\ x \in \mathbb{R}^N, \tag{1.4}$$

and $f \in L^q(\mathbb{R}^N)$, $q \in [1,\infty)$ or $f \in C_0(\mathbb{R}^N)$. For $f \in \mathcal{S}'(\mathbb{R}^N)$, $e^{t\Delta} f$ is defined by duality. A mild solution of the problem (1.1)–(1.2) is a solution of the integral equation

$$u(t) = e^{t\Delta}\varphi + a \int_0^t e^{(t-s)\Delta} \left(|\cdot|^{-\gamma} |u(s)|^\alpha u(s)\right) ds, \tag{1.5}$$

and it is in this form that we consider problem (1.1)–(1.2).

We first consider local well-posedness for the integral equation (1.5). To our knowledge, there is only one previous result of this type, Wang [15], who works in the space $C_B(\mathbb{R}^N)$ of continuous bounded functions. For $N \ge 3$, $a > 0$ and $\gamma < 2$, he proves local existence of solutions to (1.5) in $C([0,T]; C_B(\mathbb{R}^N))$ for all $\varphi \in C_B(\mathbb{R}^N)$. See [15, Theorem 2.3, p. 563].

In this paper, we prove local well-posedness in $C_0(\mathbb{R}^N)$, the space of continuous functions vanishing at infinity, and in $L^q(\mathbb{R}^N)$ for certain values of q. We also require the condition $\gamma < 2$, and in fact $0 < \gamma < 2$. Throughout the paper we put, for $\alpha > 0$, $0 < \gamma < 2$,

$$q_c = \frac{N\alpha}{2-\gamma}. \tag{1.6}$$

The critical exponent q_c plays a crucial role in this theory. We will say that q is subcritical, critical or supercritical, according to whether $1 \le q < q_c$, $q = q_c$ or $q > q_c$. We have obtained the following results.

Theorem 1.1 (*Local Well-Posedness*)**.** *Let $N \ge 1$ be an integer, $\alpha > 0$ and γ such that*

$$0 < \gamma < \min(2, N). \tag{1.7}$$

Let q_c be given by (1.6). Then we have the following.

(i) *Eq. (1.5) is locally well-posed in $C_0(\mathbb{R}^N)$. More precisely, given $\varphi \in C_0(\mathbb{R}^N)$, then there exist $T > 0$ and a unique solution $u \in C\big([0,T]; C_0(\mathbb{R}^N)\big)$ of (1.5). Moreover, u can be extended to a maximal interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t\to T_{\max}} \|u(t)\|_\infty = \infty$.*

(ii) *If q is such that*

$$q > \frac{N(\alpha+1)}{N-\gamma}, \qquad q > q_c \quad \text{and} \quad q < \infty,$$

then Eq. (1.5) is locally well-posed in $L^q(\mathbb{R}^N)$. More precisely, given $\varphi \in L^q(\mathbb{R}^N)$, then there exist $T > 0$ and a unique solution $u \in C\big([0,T]; L^q(\mathbb{R}^N)\big)$ of (1.5). Moreover, u can be extended to a maximal interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t\to T_{\max}} \|u(t)\|_q = \infty$.

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